Large time control and turnpike properties for wave equations

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Abstract

In the last decades mathematical control theory has been extensively developed to handle various models, including Ordinary and Partial Differential Equations (ODE and PDE), both of deterministic and stochastic nature, discrete and hybrid systems.

However, little attention has been paid to the length of the time horizon of control, which is necessarily long in many applications, and to how it affects the nature of controls and controlled trajectories. The *turnpike property* refers precisely to those aspects and stresses the fact that, often, optimal controls and trajectories, in long time intervals, undergo some relevant asymptotic simplification property ensuring that, during most of the time-horizon of control, optimal pairs remain close to the steady-state optimal one.

Due to the intrinsic finite velocity of propagation and the oscillatory nature of solutions of the free wave equation, optimal controls for waves are typically of oscillatory nature. But, despite this, as we shall see, under suitable coercivity conditions on the cost functional to be minimised and when controllability holds, the turnpike property is also fulfilled for the wave equation.

When this occurs, the approximation of the time-depending control problem by the steady-state one is justified, a fact that is often employed in applications to reduce the computational cost.

We present some recent results of this nature for the wave equation and other closely related conservative systems, and discuss some other related issues and a number of relevant open problems that arise in this field.

Keywords: Waves, optimal control, controllability, long time horizons, the turnpike property

1 Problem formulation

Optimal control problems play a key role in many fields and applications to industry, technology and other sciences. In the last decades the mathematical theory has been extensively developed to handle these problems for various models, including Ordinary and Partial Differential Equations (ODE and PDE), both of deterministic and stochastic nature, discrete and hybrid systems.

The existing theory provides systematic methods to prove the existence of optimal controls, characterise them through optimality conditions, build feedback controllers and efficient computational methods. Often times, however, little attention is paid to the length of the time horizon of control, which is necessarily long in many applications, and how it affects the nature of controls and controlled trajectories.

But, dealing with long time intervals is sometimes necessary to face specific applications and it has important consequences on the nature of the controls and increases the computational cost

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significantly, becoming often prohibitive. It is therefore natural to develop specific tools conceived to deal with control problems in long horizons of time, adapted to its specific nature and structure.

In this paper we summarise the recent work of our team in this field focusing on the paradigmatic example of the wave equation, and establish links between the property of *turnpike* and the classical notion of controllability.

Turnpike refers to the fact that optimal controls and trajectories, in long time intervals, undergo some relevant, and somehow unexpected, asymptotic simplification property ensuring that, during most of the time-horizon of control, optimal pairs remain close to the steady-state optimal one, the turnpike state.

Due to the intrinsic finite velocity of propagation and the oscillatory nature of solutions of the free wave equation, optimal controls for waves are typically of oscillatory nature. Thus, the fullfilement of the turnpike property might seem surprising in a first glance, although it is not really so since it is closely related to the exponential decay of solutions of the Linear Quadratic Regulator Problems (LQR) in infinite horizons of time.

Despite of this, as we shall see, under suitable coercivity conditions on the cost functional to be minimised, when controllability holds, the turnpike property is also fulfilled.

When this occurs, the approximation of the time-depending control problem by the steady-state one, a fact that is often employed in applications to reduce the computational cost, is justified.

We present some recent results of this nature for the wave equation and discuss some other related issues and a number of relevant open problems that arise in this field.

To do it on a concrete model example we mainly consider the wave equation

$$\begin{cases} y_{tt} - \Delta y = u\chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0) = y_0\,,\ y_t(0) = y_1, & \end{cases} \tag{1.1}$$

in a bounded domain Ω of the d-dimensional Euclidean space, with a control function $u = u(x,t)$ acting on an open non-empty subset ω of Ω during the time horizon $(0,T)$.

This is one of the most paradigmatic examples of control problem for an infinite-dimensional conservative model, the wave equation, arising in a variety of applied contexts. The state $y = y(x,t)$ may represent an acoustic signal or the deformation of some flexible membrane, and the control $u = u(x,t)$, whose action is localised in ω, letting vibrations to propagate freely outside ω, models some exterior applied source or force.

In this article we shall mainly focus on this model but most of the contents apply in a much broader context of conservative infinite-dimensional dynamics. We refer to [37] for a general discussion of the corresponding abstract semigroup setting.

When $(y_0,y_1)\in H_0^1(\Omega)\times L^2(\Omega)$ and $u\in L^2(\Omega\times(0,T))$ system (1.1) has a unique finite energy solution $y\in C\left([0,T];H_0^1(\Omega)\right)\cap C^1\left([0,T];L^2(\Omega)\right)$.

To motivate the problem under consideration and discriminate the various scenarios in which the *turnpike* phenomenon emerges or not, we begin considering the classical controllability problem where the control is aimed to drive the state to a given target in the final time $t = T$, paying special attention to the behaviour of controls and controlled dynamics in long time intervals.

Often in practice, when the free dynamics of the system under consideration enjoys some stability property so that, for instance, time-evolving solutions tend to steady-state ones as $t\to\infty$, it is expected that time evolving controls will also reproduce that property when the time control horizon is long enough. This is the so-called *turnpike property* and it is very much in agreement with intuition, in particular, for parabolic-like problems, where the inner dissipative mechanisms are likely to force the desired stability property of controls and controlled dynamics.

But whether this behaviour is still to be expected for wave-like models, where solutions of the free dynamics are of oscillatory nature, conserve energy and, therefore, do not enjoy the property of asymptotic simplification, is less clear.

In this paper we shall however show, collecting previous earlier results, that the turnpike property still holds for wave-like models under suitable controllability assumptions. This fact is relevant since even if, a priori, one is simply interested in other kinds of optimal control problems, such as minimising a quadratic cost functional, without paying attention to the terminal conditions at the final time $t = T$, the emergence of the turnpike phenomenon is very tightly connected with the fulfilment of the controllability property.

These results are relevant and find application in various different ways:

1. The turnpike property ensures the asymptotic simplification of controls and controlled trajectories for optimal control problems involving wave-like models in long-time horizons, so that optimal pairs are close to the steady-state ones during most of the time interval. More precisely, except for an initial time layer $[0, \tau]$ and a final one $[0, T - \tau]$, in which the controlled dynamics has to match the initial and terminal conditions, during the rest of the time interval $[\tau, T - \tau]$ the control and controlled trajectories are exponentially close to the steady-state ones.

2. This property can be used as test for the accuracy of the numerical simulation codes: those not reproducing the property of turnpike could be considered as unsuitable ones.

3. This asymptotic simplification property can serve also to initialise iterative methods for solving the optimality system characterising optimal pairs, employing the steady-state optimal pairs, which are cheaper to compute, in the initialisation step.

4. The turnpike state also serves as an initialisation to Receding Optimal Control (ROC) or Model Predictive Control (MPC) methods (see [23], [24]).

As we shall see in the present paper, even when dealing with classical Linear-Quadratic (LQ) optimal control problems, for the turnpike property to hold two ingredients will be needed:

1. The system under consideration needs to enjoy the property of controllability. And this is so even if, a priori, one is not interested in controllability issues. But for the turnpike property to hold, ensuring in particular that the optimal trajectory stays most of the time near the turnpike point, one needs that the action of the control suffices to ensure the controllability property.

2. The cost functional needs to be sufficiently coercive. In other words, it needs to penalise the control but also the state sufficiently so that the partial information of the state involved in the cost functional suffices to get complete information in the full state. This is a observability property that, for wave-like equations, requires some geometric restrictions on the subdomain where the control is being applied and a sufficiently long control time-horizon. The later is not an issue when analysing turnpike properties since the time-horizon tends to infinity. But the former needs to be taken into account with care, imposing suitable geometric conditions on the support of the control.

In order to have a complete understanding of the turnpike property one needs to carefully analyse the behaviour of the Optimality System (OS) characterising optimal controls and trajectories. It is constituted by a coupled system of two wave equations, one evolving in the forward sense of time while the other one does it in the backward sense. State and control are fully coupled through this system. Accordingly, the turnpike property needs to hold in both variables (state and adjoint state or co-state) simultaneously.

The two conditions under which the turnpike property mentioned above holds play dual and complementary roles in what concerns the behaviour of the OS to ensure the turnpike property. The controllability of the state equation can be understood as an observability property on the adjoint. On the other hand, the fact that the cost functional involves enough information on the state can be understood as an observability property of the state equation and thus, a controllability one for the adjoint system.

Overall, under these assumptions, the turnpike property is fulfilled and during most of the time-horizon $[0, T]$ optimal controls and trajectories are exponentially close to the steady state one. The later ones are, of course, much easier to compute since they are determined as the solution of an optimal control problem for the underlying elliptic equation, namely

$$\begin{cases} \Delta y = u\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \Omega. \end{cases} \tag{1.2}$$

In a first instance, this fact could seem counterintuitive because of the purely oscillatory nature of the wave equation in the absence of damping and the fact that the spectral complexity of the initial data is preserved along the free dynamics. These two properties could prevent us from expecting such simplification to occur on controlled trajectories. But this does indeed occur, similarly as the solutions of LQR problems in infinite time horizons experience the property of exponential stabilisation. As we shall see, somehow, in long time intervals, the optimal controls (according to the cost criteria fulfilling the conditions above) not only control the system but they also damp it, ensuring thus the stability or stabilisation property of turnpike.

As we shall see, to a large extent, this property is closely related to the well known property that controllability implies stabilisation (see e.g. [28] for a simple formulation sufficient to deal with the specific example under consideration), which is the dual of the classical's Russell's principle ensuring that "stabilisation implies controllability" ([43]). The turnpike property could be added to this list of properties that are nearly equivalent in long time horizons.

We could summarise the content of this paper by saying that not only controllability implies stabilisation but also the turnpike property for a large class of cost functionals, provided they are coercive enough to yield full information on the state of the system.

The rest of this paper is organised as follows.

In the next Section we briefly discuss some of the existing turnpike literature. In Section 3 we recall some preliminary well known material about the properties of controllability, stabilisation and observation of the wave equation. In Section 4 we discuss the classical controllability problem for the wave equation that seems to manifest the lack of turnpike property. In Section 5 we show however that turnpike occurs provided the choice criterion for the control controlling the system is adequately modified. In Section 6 we summarise the theory developed in [37] that explains in which conditions the turnpike property occurs and allows clarifying the two opposite behaviours described in previous sections, and explaining how they are both compatible. In Section 7 we discuss some of the existing literature on the subject and present some open problems.

2 Bibliographical comments

There is by now an extensive literature on the turnpike property. Here we comment some of the published papers in this topic. The bibliography at the end of this paper contains a non complete list of other articles and books devoted to this, or closely related topics.

The turnpike property of optimal trajectories was introduced in the context of finite-dimensional discrete-time optimal control problems in Economy (see, e.g., [35], [16], [44]).

In the last decades there has been a growing interest and a significant number of papers was published in this area, both for time-discrete and time-continuous finite dimensional systems (see, for instance, [2, 3, 9, 10, 23, 16, 20, 25, 27, 33, 35, 37, 38, 39, 49, 58, 59, 60] and references therein).

The problem of turnpike for abstract linear finite and infinite-dimensional systems has been considered in [37] where the specific example of heat and wave equations has also been analysed, obtaining exponential turnpike properties and extending previous works as [13]. In particular, in [37], the wave equation was shown to fulfil the exponential turnpike property provided the control satisfies the so-called Geometric Control Condition (GCC) (see [64]), ensuring that all rays of Geometric Optics enter the control subdomain in an uniform time. The particular case of the linear $1-d$ wave equation was treated in more detail in [27].

In [37] it was also shown that, even if the control subdomain ω does not fulfil the GCC, a weak version of the turnpike property is satisfied, ensuring a logarithmic convergence rate of the optimal trajectories towards the steady-state one. This is a typical infinite-dimensional phenomenon that cannot hold in the linear finite-dimensional context. It is due to the fact that if GCC fails, the wave equation is still controllable, but in a much smaller space (see [5]). The cost of controlling high frequencies grows exponentially as the frequency tends to infinity, and this is responsible for the turnpike property to emerge slowly, with a logarithmic rate as $T \to \infty$. This is in agreement with the well-known results on the logarithmic decay rates of the energy of smooth solutions of the damped wave equation, when the damping is effective on subdomains that do not satisfy the GCC.

The work in [37] was inspired in [7], [8] where the long-time averages of solutions for backward-forward mean field game systems were analysed, independently, without reference to the turnpike theory.

The turnpike property has been also the object of intensive investigation in the nonlinear finite-dimensional setting (see, for instance, the list of references in the recent paper [49]). The main ingredient in [49] is an exponential dichotomy transformation established in [53] to uncouple the two-point boundary value problems arising from the Pontryagin maximum principle, reflecting the hyperbolic nature of the Hamiltonian system, as introduced in [42].

The notion of turnpike towards a steady state, a manifestation of the tendency of optimal pairs to undergo some asymptotic simplification property in long time intervals, can be extended in various ways. For instance, in [46], we deal with the problem of periodic turnpike, also considered in the recent paper [56] within the dissipativity context, making use of the hyperbolicity properties of the Optimality System. See also [4], [21].

The results in [37] have been also extended to semilinear heat equations in [38] and to the $2-d$ Navier-Stokes equations in [55] but, for that to be done, a smallness condition was assumed in the target. This is very likely a purely technical condition. Dealing with semilinear problems and large targets and deformations is a widely open problem that we shall discuss below in some more detail.

The property of turnpike for shape design problems for the $2d$ heat equation has been considered in [48], for time-independent shapes. The extension of the results of [48] to shape design problems where shapes are also allowed to depend in time is an interesting and challenging open problem. The situation is similar in the context of design problems where the control is the diffusivity coefficient, see [1]. Similar questions arise in different problems arising in Inverse Problem theory as well ([12], [29]).

In the last section we discuss some possible extensions and open problems inspired on this rich earlier literature and the developments presented in this paper.

3 Preliminaries on controllability, observability and stabilisation

The problem of controllability consists roughly in *describing the set of reachable final states* for all possible solutions of the wave model under consideration,

$$R\left(T;(y_0,y_1)\right)=\left\{(y(T),y_t(T)):u\in L^2(\Omega\times(0,T))\right\}$$

and building controls that efficiently drive the system to the final reachable states.

This set is the affine subspace of the final states that the solutions reach at time $t=T$, starting from the initial datum (y_0,y_1), when the control u varies all over $L^2(\Omega\times(0,T))$. Note however that the action of the control is localised in ω. Thus, the controls may also be viewed to belong to the more limited class $L^2(\omega\times(0,T))$.

One may distinguish various notions or degrees of controllability. *Exact controllability* consists on analysing whether

$$R\left(T;(y_0,y_1)\right)=H_0^1(\Omega)\times L^2(\Omega)$$

for all $(y_0,y_1)\in H_0^1(\Omega)\times L^2(\Omega)$, i. e. whether the system can be driven from any finite energy initial configuration to any final one by a suitable choice of the control.

Since system (1.1) is linear and time-reversible, a necessary and sufficient condition for exact controllability to hold is whether $(0,0)\in R\left(T;(y_0,y_1)\right)$ for all $(y_0,y_1)\in H_0^1(\Omega)\times L^2(\Omega)$. This is the so called *null-controllability* property in which the goal is reduced to

$$y(T)\equiv y_t(T)\equiv 0. \tag{3.1}$$

Since we are dealing with solutions of the wave equation, due to the finite speed of propagation, for controllability to hold, the control time T has to be sufficiently large, the trivial case in which the control subdomain ω coincides with the whole domain Ω being excepted.

More precisely, the so-called *Geometric Control Condition (GCC)* (see [64]), ensuring that all rays of Geometric Optics enter in time T the control subdomain ω, is a sharp sufficient condition for controllability. Under this condition the controllability property holds.

This sharp geometric condition on the control subdomain is also related with the decay properties of the damped equation and, more generally, with the *exponential stabilisability* of the damped wave equation:

$$\begin{cases} y_{tt}-\Delta y+y_t\chi_\omega=0 & \text{in } (0,\infty)\times\Omega\\ y=0 & \text{on } (0,\infty)\times\partial\Omega\\ y(0)=y_0\,,\; y_t(0)=y_1. \end{cases} \tag{3.2}$$

In fact, if $\omega\subset\Omega$ satisfies the GCC, then the solutions of (3.2) satisfy

$$E(t):=\int_\Omega [y_t(t)^2+|\nabla y(t)|^2]dx\leq Ce^{-\mu t}E(0) \qquad \forall t>0$$

for some C, $\mu>0$ independent of the solution, where the energy $E(t)$ is defined as

$$E(t)=\frac{1}{2}\int_\Omega [|\nabla y(x,t)|^2+|y_t(x,t)|^2]dx.$$

This means that, in those circumstances in which the controllability property holds, the feedback control

$$u=-y_t,$$

localised in the control subset ω, ensures the property of exponential decay.

This is the first manifestation of the link between the finite-time controllability property of the system and the asymptotic properties as $t\to\infty$. This equivalence was noticed by D. Russell [43] who also indicated how, out of the exponential decay property of the damped system, one can build a control ensuring controllability in finite time, as the limit of an iterative process. This fact has also been employed recently to develop efficient numerical approximation methods (see [50] and [19]), implementing the iterative algorithm introduced in [43] but applied to a suitable discrete approximation of the PDE.

Both properties, controllability and exponential decay of the damped system, in the present setting, are equivalent to the so-called *observability* property for the adjoint system:

$$\begin{cases} \varphi_{tt}-\Delta\varphi=0 & \text{in } (0,T)\times\Omega\\ \varphi=0 & \text{on } (0,T)\times\partial\Omega\\ \varphi(T)=\varphi_0\,,\; \varphi_t(T)=\varphi_1. \end{cases} \tag{3.3}$$

This observability property ensures the existence of a constant $C > 0$ which, of course, depends on Ω, ω and $T > 0$, but is independent of the adjoint solution under consideration, such that

$$||\varphi_0||^2_{L^2(\Omega)} + ||\varphi_1||^2_{H^{-1}(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \tag{3.4}$$

for all solutions of (3.3).

This observability inequality holds if and only if the GCC is satisfied (some exceptional cases being excluded involving glancing rays).

Observability not only implies the exponential decay for the damped system (3.2) as proved by [28]. In fact, out of the observability inequality one can build more general feedback operators ensuring the exponential decay of the corresponding damped system at any exponential rate (see [30]).

There is an extensive literature related to the observability and controllability of wave-like equations. Due to the purely conservative and infinite-dimensional character of the model some of the classical features of optimal control problems such as the bang-bang control property for L^∞-controls in minimal time may fail (see for instance [26]). See also the monographs [51] and [54] for a general presentation of the topic.

4 The apparent lack of turnpike property

As pointed out by J. L. Lions [32], when the observability inequality holds, controllability holds as well, and there is a unique control achieving the terminal condition of minimal $L^2(\omega \times (0,T))$-norm. More precisely, if the goal of the control system is to drive the system from the initial state (y_0, y_1) to the trivial one in time $t = T$, i.e.

$$(y(x,T), y_t(x,T)) \equiv (0,0),$$

whenever this is possible, the control u of minimal L^2-norm can be characterised as the restriction to ω of a distinguished solution of the adjoint system (3.3)

$$u = \tilde{\varphi} \tag{4.1}$$

where $\tilde{\varphi}$ is the solution of (3.3) with data $(\tilde{\varphi}_0, \tilde{\varphi}_1)$ where $(\tilde{\varphi}_0, \tilde{\varphi}_1)$ is the minimiser of the functional

$$K(\varphi_0, \varphi_1) = \frac{1}{2} \int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega [y_0 \varphi_t(x,0) - y_1(x)\varphi(x,0)]dx \tag{4.2}$$

in $L^2(\Omega) \times H^{-1}(\Omega)$.

The observability property of the system plays a key role when ensuring the coercivity of the functional K and, thus, to guarantee the existence of a minimiser.

The control given by (4.1), in view of the very properties of the conservative adjoint wave equation, has a periodic or quasi-periodic nature in time. For instance, in the $1d$ domain $\Omega = (0, L)$, the control is time-periodic of period $2L$.

This is in agreement with intuition: Solutions of the wave equation are of an oscillatory nature, and controls should behave the same way. The control of minimal $L^2(\omega \times (0,T))$-norm enjoys this very property, indeed.

As we have seen, both the exponential decay of the damped wave equation or its controllability require achieving an observability estimate of the form (3.4), which is linked to the GCC. And proving these observability inequalities needs significant developments, using tools from the theory of non-harmonic Fourier series, microlocal analysis, multipliers or Carleman inequalities.

In practice, when addressing control problems and, in particular, when looking for numerical approximations, it is natural to address the controllability problem from the point of view of optimal control through a penalisation procedure.

The classical optimal control approach consists on minimising a cost functional of the form:

$$J(u) = \frac{1}{2}\left[||y(T)||^2_{H^1_0(\Omega)} + ||y_t(T)||^2_{L^2(\Omega)} + \int_0^T\int_\omega u^2 dxdt\right], \tag{4.3}$$

which penalises both the size of the control and also the final value of the solution at time $t = T$.

In order to enhance the target $y(T) \equiv y_t(T) \equiv 0$, one can rather consider a penalised functional of the form

$$J_N(u) = \frac{1}{2}\left[N||y(T)||^2_{H^1_0(\Omega)} + N||y_t(T)||^2_{L^2(\Omega)} + \int_0^T\int_\omega u^2 dxdt\right], \tag{4.4}$$

with $N > 0$ large enough.

Increasing N emphasises the goal of making the final energy of the solution at time $t = T$ to be smaller and smaller, so that the state (y, y_t) approaches the target $(0, 0)$ at the final time $t = T$.

Classical results from the Calculus of Variations, ensuring that a convex and coercive functional in a Hilbert space achieves its minimum, suffice to guarantee the existence of a minimiser $u_N \in L^2(\omega \times (0, T))$ for the functional J_N, without any observability or controllability property. Furthermore, uniqueness is assured by the strict convexity of the functional.

By the contrary, in order to show that the minimisers u_N are uniformly bounded in $L^2(\omega\times(0, T))$ as N tends to infinity, one needs the controllability property to be fulfilled. In that case, it can be shown that the limit of u_N in $L^2(\omega\times(0, T))$ as N tends to infinity leads to the desired exact control.

Accordingly, in practice, minimising J_N is not only a means of approximating the exact control when it exists, but also a way of testing the controllability properties of the system through the boundedness criterion of the sequence of minimisers u_N.

Let us now analyse the structure of the controls we achieve when minimising functionals of the form J_N. To fix ideas and simplify the presentation let us simply consider $N = 1$, going back to the functional J in (4.3).

The Optimality System (OS) characterising the control u minimising the functional J can also be characterised as

$$u = \psi \quad \text{in } \omega \times (0, T) \tag{4.5}$$

where ψ is a distinguished solution of the adjoint system coupled together with the state equation in the following manner:

$$\begin{cases} y_{tt} - \Delta y = \psi\chi_\omega & \text{in } (0, T) \times \Omega \\ y = 0 & \text{on } (0, T) \times \partial\Omega \\ y(0) = y_0\,,\; y_t(0) = y_1, & \text{in } \Omega \\ \psi_{tt} - \Delta\psi = 0 & \text{in } (0, T) \times \Omega \\ \psi = 0 & \text{on } (0, T) \times \partial\Omega \\ \psi(T) = -\,y_t(T)\,,\; \psi_t(T) = -\,\Delta y(T) & \text{in } \Omega. \end{cases} \tag{4.6}$$

System (4.6) is constituted by the coupling of the state and the adjoint system, one with initial conditions at $t = 0$ and the other one with terminal conditions at $t = T$. Thus the system does not enter into the class of classical Cauchy problems. The existence of a unique solution for (4.6) is a consequence of the fact that it characterises the unique critical point, the minimiser, of J that we know it exists.

We observe that, once again, the control preserves the time-periodic or quasi-periodic structure of the solutions of the wave equation since it is actually given by (4.5), as the restriction to the control set ω of the solution ψ of the adjoint system, which is the free wave equation. In particular, because of the conservative character of the wave equation, the energy of ψ, which, in the appropriate functional setting has to be defined as

$$E_-(t) = \frac{1}{2}\left(||\psi(t)||^2_{L^2(\Omega)} + ||\psi_t(t)||^2_{H^{-1}(\Omega)}\right), \tag{4.7}$$

is time independent.

This fact does not depend on the value of the penalty parameter N and the length of the time horizon T. We thus observe, in particular, that by taking the time horizon for control $[0,T]$ to be longer, the control preserves its oscillatory nature and does not experience any asymptotic simplification. In particular, the controls and controlled trajectories, even when the time horizon for control is very long, do not present the property of turnpike of being close to a steady configuration most of the time.

5 The emergence of the turnpike property

One could think that this fact, the lack of turnpike property, is unavoidable, purely related to the inner oscillatory nature of the solutions of the wave equation. But, according to our previous work in the subject, as we shall see, this is not a correct interpretation since the behaviour of optimal controls and trajectories in long time intervals, and whether, more precisely, the turnpike property holds or not, depends heavily on the chosen criterion to characterise the optimal control.

We can for instance, rather than dealing with the minimisation of the functional J_N, consider the perturbed quadratic functional

$$\tilde{J}_N(u) = \frac{1}{2}\left(N||y(T)||^2_{H^1_0(\Omega)} + N||y_t(T)||^2_{L^2(\Omega)} + \int_0^T \int_\omega u^2 dxdt + \int_0^T \int_\Omega |\nabla y|^2 dxdt \right). \tag{5.1}$$

This quadratic functional involves three terms. The first and second one, as in the functionals above, have as goal driving the system to equilibrium at the final time $t = T$ and keeping the size of the control limited, respectively. However, in the third term of this new functional, the size of the solution all along the trajectory is penalised as well.

The existence of a unique minimiser $\tilde{u}_N$ for $\tilde{J}_N$ is again an obvious fact and, in the limit as N tends to infinity, it will lead to a control $\tilde{u}$ controlling exactly the solution of the wave equation to the null state in time $t = T$ as well. But, of course, this new control will not coincide with the previous one since it has been characterised as the minimiser of a different functional. But this does not generate any contradiction since the control controlling a system exactly is not necessarily unique.

The Optimality System (OS) characterising the control $\tilde{u}_N$ varies with respect to (4.6). Although the control is still given by (4.5), the corresponding OS takes the form:

$$\begin{cases} y_{tt} - \Delta y = \psi \chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0) = y_0\,,\ y_t(0) = y_1, & \text{in } \Omega \\ \psi_{tt} - \Delta\psi = \Delta y & \text{in } (0,T)\times\Omega \\ \psi = 0 & \text{on } (0,T)\times\partial\Omega \\ \psi(T) = -N\,y_t(T)\,,\ \psi_t(T) = -N\,\Delta y(T) & \text{in } \Omega. \end{cases} \tag{5.2}$$

In this case we see that, even if the control is given as the restriction to ω of ψ, ψ is not longer the solution of the free wave equation. Rather the wave equation satisfied by ψ is forced by a right hand side term involving the state y, namely Δy. In order to analyse the nature of the control, both equations, the state and the adjoint one, need to be handled simultaneously because of their coupling.

The corresponding system, ignoring the initial and terminal conditions reads:

$$\begin{cases} y_{tt} - \Delta y = \psi \chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ \psi_{tt} - \Delta\psi = \Delta y & \text{in } (0,T)\times\Omega \\ \psi = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{5.3}$$

In order to analyse the behaviour of the solutions of this system it is worth to consider the simplest case in which $\omega = \Omega$, i.e. the control acts everywhere in the domain ω where waves propagate, in which case, of course, the GCC is fulfilled, and the controllability property holds, together with the observability one and the exponential decay of the energy of the corresponding damped wave equation.

In this particular case the system reads:

$$\begin{cases} y_{tt} - \Delta y = \psi & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ \psi_{tt} - \Delta\psi = \Delta y & \text{in } (0,T)\times\Omega \\ \psi = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{5.4}$$

In order to analyse the behaviour of its solutions it is convenient to perform a Fourier series decomposition. Let $\{\phi_j\}_{j\geq 1}$ and $\{\lambda_j\}_{j\geq 1}$ be the eigenfunctions and the eigenvalues of the Dirichlet Laplacian,

$$\begin{cases} -\Delta\phi_j = \lambda_j\phi_j & \text{in } \Omega \\ \phi_j = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

the eigenfunctions $\{\phi_j\}_{j\geq 1}$ being normalised to constitute an orthonormal basis of $L^2(\Omega)$.

Then the pair of solutions of (5.4) can be written as:

$$(\psi, y) = \sum_{j\geq 1} (\hat{\psi}_j, \hat{y}_j)\exp(\mu_j t)\phi_j,$$

for suitable time frequencies μ_j and Fourier coefficients $(\hat{\psi}_j, \hat{y}_j)_{j\geq 1}$.

It is easy to see that these coefficients are linked by the relations

$$\mu^2 + \lambda = \frac{\hat{\psi}}{\hat{y}}, \quad \mu^2 + \lambda = -\lambda\frac{\hat{y}}{\hat{\psi}}. \tag{5.6}$$

Here and in the sequel we have omitted the index j in all the terms entering in the system to simplify the notation.

In view of this, we have

$$[\mu^2 + \lambda]^2 = -\lambda$$

and, accordingly,

$$\mu = \pm\sqrt{-\lambda \pm i\sqrt{\lambda}}. \tag{5.7}$$

We observe that μ takes four different complex values, two pairs of conjugate ones actually. Two of them have strictly positive real parts, while the two others negative ones.

This means that the solutions of (5.4) are constituted by the superposition of two time evolving components of oscillatory nature, one decaying exponentially as $t\to\infty$ while the other one grows exponentially.

This behaviour is compatible with the turnpike property, according to which the solution should be close to the optimal steady state during most of the time horizon of control $[0,T]$, when T is large.

The steady state optimal configuration can be characterised as the solution of the corresponding steady optimal control problem. It concerns the steady equation

$$\begin{cases} -\Delta z = v\chi_\omega & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.8}$$

the optimal control $z = z(x)$ being given as the minimum of the steady cost functional

$$\tilde{J}_s(v) = \frac{1}{2}\left(\int_\Omega |\nabla z|^2 dx + \int_\omega v^2 dx\right), \tag{5.9}$$

which is independent of the penalisation factor N.

Obviously, the existence and uniqueness of the minimiser (z, v) for this functional in $L^2(\omega)$ is straightforward. In fact the minimiser, in this case, is the trivial one, i. e. $v \equiv 0$, leading to the trivial solution $z \equiv 0$. But similar arguments would apply if the minimiser for the steady problem was non-trivial.

The turnpike property ensures that the existence of positive constants $C > 0, \alpha > 0$ and T_0 such that, for all $T > T_0$, the solution (y, u) of the time-evolution control problem minimising the functional $\tilde{J}_N$ is close to the steady-state one in the sense that, for all $t > 0$,

$$||y(t)-z||_{H_0^1(\Omega)}+||y_t(t)||_{L^2(\Omega)}+||u(t)-v||_{L^2(\omega)} \leq C[\exp(-\alpha t)+\exp(-\alpha(T-t))][||y_0-z||_{H_0^1(\Omega)}+||y_1||_{L^2(\Omega)}]. \tag{5.10}$$

Recall that, in the present particular case, $z \equiv 0$.

This guarantees that both the control and the state stabilise around the steady state ones away from $t = 0$ and $t = T$ for T large enough.

The situation therefore is rather different to the one we encountered when analysing the controls minimising the original functional J_N. In that case, for $\omega = \Omega$, the OS would have been

$$\begin{cases} y_{tt} - \Delta y = \psi & \text{in } (0,T) \times \Omega \\ y = 0 & \text{on } (0,T) \times \partial\Omega \\ \psi_{tt} - \Delta \psi = 0 & \text{in } (0,T) \times \Omega \\ \psi = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases} \tag{5.11}$$

The Fourier decomposition, in view of the fact that ψ fulfils the free wave equation, would then lead to

$$\mu^2 + \lambda = 0. \tag{5.12}$$

Accordingly, μ would be purely imaginary, thus leading to solutions of purely oscillatory nature, in agreement with the previous observations. In this case, in particular, the adjoint state ψ and, accordingly, the control u, would have a purely oscillatory behaviour without never stabilising around the steady state one.

We conclude that:

1. The turnpike property may hold not only for control problems in which the free underlying dynamics has in itself the property of asymptotic simplification. It may also take place for wave-like equations.

2. This depends heavily on the fact that both the state and the control are penalised on the cost functional to be minimised, as for instance in (5.1).

3. For this to hold the system under consideration needs to enjoy the property of controllability.

The fact that penalising the state in the functional to be minimised affects the turnpike property is very easily seen in the previous example. Otherwise, the adjoint state is independent of the state, satisfies a free wave equation and does not experience the property of asymptotic simplification or turnpike.

But the third fact, according to which controllability contributes to the turnpike property of optimal control problems is less obvious and can be understood in the following manner.

Even if, in principle, the control is entering into a conservative dynamics as

$$y_{tt} - \Delta y = u\chi_\omega, \quad \text{in } (0,T) \times \Omega,$$

the fact that the system under consideration is controllable, i. e. ω satisfies the GCC, allows viewing the solution y as that of the damped system

$$y_{tt} - \Delta y + y_t\chi_\omega = (u + y_t)\chi_\omega, \quad \text{in } (0,T) \times \Omega.$$

And this new damped system has the property of exponential decay in the absence of control. This passage from the conservative to the exponentially damped dynamics is ensured by an additive translation of the control from u into $w = u + y_t$. Under the controllability assumption we can thus assume that we are dealing with a control problem for an exponentially stable dynamics. The cost functional to be minimised being coercive in the original state and control variables (y, y_t) and u, then it remains coercive when written in the new control variable w as well.

In a first reading, it could also be argued that the functionals above are coercive on the state itself y but not on the velocity variable y_t, which is also part of the state. Thus, strictly speaking, for the functional to be coercive on the full state (y, y_t) in $H_0^1(\Omega) \times L^2(\Omega)$, the cost functional should also involve a term of the form

$$\int_0^T \int_\Omega |y_t|^2 dxdt.$$

But, as we have seen above, this is not strictly necessary. Penalising the state y in $H_0^1(\Omega)$ suffices and this is due to the classical property of equipartition of energy that guarantees that, along a given time interval $[0, T]$, the energy concentrated on the y and y_t components is nearly the same, in $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively, up to a compact reminder term.

In the presentation above we have penalised both the control and the state so as to keep them as small as possible in the corresponding norms. But the same could be done aiming that the solution remains close to another reference state. For instance, the functional $\tilde{J}_N$ in (5.1) could be slightly modified to consider

$$\tilde{J}_{N,\bar{y}}(u) = \frac{1}{2}\left(N||y(T)||^2_{H_0^1(\Omega)} + N||y_t(T)||^2_{L^2(\Omega)} + \int_0^T \int_\Omega |\nabla y - \nabla \bar{y}|^2 dxdt + \int_0^T \int_\omega u^2 dxdt\right), \tag{5.13}$$

$\bar{y}$ being any element of $H_0^1(\Omega)$.

When minimising the functional we penalise the state at $t = T$ so to make it as close as possible to $(0, 0)$ with a control of minimal norm but, simultaneously, keeping the trajectory $y(t)$ close to the reference target $\bar{y}$. The turnpike property holds in this situation too but, in this case, of course, the corresponding steady-state problem has to be modified accordingly to deal with the functional

$$\tilde{J}_s(v) = \frac{1}{2}\left(\int_\Omega |\nabla z - \nabla \bar{y}|^2 dx + \int_\omega v^2 dx\right). \tag{5.14}$$

The turnpike property ensures, once again, that in long time control horizons optimal controls and controlled trajectories remain exponentially close to the steady-state ones during most of the time interval $[0, T]$ as in (5.10).

Remark 1. As we shall see below, the turnpike property does not need to involve a cost functional that is coercive in the full state, but just on some projection ensuring the observability inequality to hold. Thus, the penalisation term

$$\int_0^T \int_\Omega |\nabla y - \nabla \bar{y}|^2 dxdt$$

in the cost functional could be replaced by

$$\int_0^T \int_{\mathcal{O}} |\nabla y - \nabla \bar{y}|^2 dxdt$$

with some subset $\mathcal{O}$ of Ω satisfying the GCC and the same turnpike property would hold.

The situation would be different however if the penalisation on the state was done in a weaker norm, for instance, the L^2-one,

$$\int_0^T \int_\Omega |y - \bar{y}|^2 dxdt.$$

In that case the exponential turnpike property would not be achieved and one would rather expect a polynomial version. The situation is similar to that of damped conservative semigroups, with compact damping operators, that fail to guarantee the exponential stabilisation property.

But a complete analysis of this issue would need further developments in the spirit of those in [37] where, to some extent, a less favourable situation was considered; that in which the control enters in a set ω that does not fulfil the GCC, making exponential stabilisation impossible. ■

6 Turnpike for abstract conservative semigroups

We now briefly present the abstract theory developed in [37] that explains the phenomena described above about the possible manifestation of turnpike phenomena. In [37] both finite-dimensional and infinite-dimensional models are considered, distinguishing parabolic-like and hyperbolic-like semigroups. Here we recall the main results in [37] devoted to abstract wave equations which suffice to explain the main observations of previous sections on the emergence of the turnpike property or the lack of it.

Consider the second order in time abstract system

$$\begin{cases} y_{tt} + Ay = Bu & \text{in } (0,T) \\ y(0) = y_0\,,\; y_t(0) = y_1\,. \end{cases} \tag{6.1}$$

We assume that X, H are Hilbert spaces such that $X \subset H \subset X'$ (with dense embedding), H being identified with its dual and $A \in \mathcal{L}(X, X')$ is a linear continuous operator. For simplicity we assume that A is coercive, i. e.

$$\langle Ay, y\rangle_{X',X} \geq \alpha ||y||_X^2,\; \forall y \in X. \tag{6.2}$$

This implies in particular that range$(A) = X'$.

We also assume that $B \in \mathcal{L}(U, H)$ and $C \in \mathcal{L}(X, V)$, where U, V, are Hilbert spaces.

Here and in the sequel we denote by $\langle \cdot, \cdot\rangle_{X,X'}$ (resp. $\langle \cdot, \cdot\rangle_{X',X}$) the canonical duality pairing between X and its dual X' (resp. X' and X).

Since we are mainly interested in wave-like equations we assume that A is self-adjoint so that the system under consideration is conservative, i.e, such that the energy

$$E(t) = \frac{1}{2}[\langle Ay(t), y(t)\rangle^2_{X,X'} + ||y_t(t)||_H^2]$$

is conserved along trajectories whenever $u \equiv 0$.

The abstract system (6.1) models various wave-like equations. The setting in [37] is slightly more general since it allows considering also models involving damping terms. But we focus here on conservative models, our goal being to describe how, despite the fact that the model under consideration is conservative, the turnpike phenomena may emerge.

Such assumptions on B and C can be relaxed (see [37]) to deal, for instance, with boundary control problems for wave-like equations. But, for the sake of simplicity, we shall assume that those operators are bounded. This allows avoiding unnecessary technical difficulties. On the other hand, this functional framework suffices to deal with the problem of the internal control of the wave equation analysed above.

Under these assumptions, for $(y_0, y_1) \in X \times H$ and $u \in L^2(0,T;U)$, system (6.1) admits a unique solution $y \in C^0([0,T];X) \cap C^1([0,T];H)$ (see e.g. [31], [Chapter IV, Thm 1.1]).

Given the target $g \in V$, we consider the optimal control problem in the time horizon $[0,T]$:

$$\min\left\{ J^T(u) = \frac{1}{2}\int_0^T \left[||u(t)||_U^2 + ||Cy(t) - g||_V^2\right] dt\,, \quad u \in L^2(0,T;U)\right\}$$

where y is the solution of (6.1).

Note that, here, for the sake of simplicity of the presentation, the value of the state at time $t = T$ is not penalised in the cost functional J^T although similar results apply when those extra terms ($||y(T)||_X^2$ and $||y_t(T)||_H^2$) are added to the cost functional. We do not impose either terminal conditions on the state at $t = T$, corresponding to the fulfillement of the controllability property. But the same analysis applies in these other cases too. Here we have preferred to limit our presentation to this functional to avoid unnecessary technical difficulties.

Similarly we consider the steady state minimisation problem

$$\min \left\{ J_s(u) = \frac{1}{2}\left[||u||_U^2 + ||Cy - g||_V^2\right], \quad u \in U, \, y \in X : \, Ay = Bu \right\} \tag{6.3}$$

Both minimisation problems, the time-evolution and the steady-state ones, have unique solutions, that we denote respectively by $(u^T(t), y^T(t))_{\{t\geq 0\}}$ and $(\bar{u}, \bar{y})$. This can be easily deduced from the fact that the functionals J^T and J_s under consideration are strictly convex, continuous and coercive in the Hilbert spaces $L^2(0, T; U)$ and U respectively. The optimal pairs $(u^T(t), y^T(t))_{\{t\geq 0\}}$ depend, obviously, on the time-horizon $[0, T]$ but, often times, we will not make this dependence explicit on the notation, except when necessary to avoid confusion.

Furthermore, taking into account that

$$J^T(u^T) = \min J^T(u) \leq J^T(0) = \frac{1}{2}\int_0^T \left[||C\tilde{y}(t) - g||_V^2\right] dt,$$

we get a first upper bound on the optimal pair $(u^T(t), y^T(t))_{\{t\geq 0\}}$, $\tilde{y}$ being the solution of uncontrolled system corresponding to $u \equiv 0$.

At this point it is important to observe that this estimate, by itself, does not guarantee the uniform boundedness of the optimal pairs $(u^T(t), y^T(t))_{\{t\geq 0\}}$ in $[0, T]$ as $T \to \infty$. Indeed, the equation under consideration being conservative the free solution $\tilde{y}$ (in the absence of control, i. e. $u \equiv 0$) does not decay and, therefore, as $T \to \infty$, $\int_0^T \left[||C\tilde{y}(t) - g||_V^2\right] dt$ may grow linearly.

At this early point of our analysis we observe the essential role that the controllability property of the system (6.1) plays, so to ensure that all optimal pairs remain uniformly bounded. To do it, rather than considering $\tilde{y}$ as the solution corresponding to $u = 0$, we take $\tilde{y}$ as being the solution of the damped system with $u = -B^*\tilde{y}_t$. In view of the fact that controllability implies the stabilisation property, this solution $\tilde{y}$ decays exponentially in the energy space. This guarantees the uniform boundedness of $\int_0^T \left[||B^*\tilde{y}_t(t)||_U^2 + ||C\tilde{y}(t)||_V^2\right] dt$ as $T \to \infty$, and consequently of the optimal pairs $(u^T(t), y^T(t))_{\{t\geq 0\}}$ when $g \equiv 0$.

Thus, as mentioned in previous sections, even if our goal was simply to analyse the asymptotic behaviour as $T \to \infty$ of the optimal pairs, the need of the controllability assumption emerges.

In fact, when the system under consideration is controllable the optimal control problem makes sense in the infinite horizon as well provided the target is $g = 0$:

$$\min \left\{ J_0^\infty(u) = \frac{1}{2}\int_0^\infty \left[||u(t)||_U^2 + ||Cy(t)||_V^2\right] dt, \quad u \in L^2(0, \infty; U) \right\}. \tag{6.4}$$

As mentioned above, the controllability property, thanks to the stabilisation, ensures that the class of admissible controls for J_0^∞ is non empty.

Concerning the steady-state problem we easily observe that

$$J_s(\bar{u}) = \min J_s(u) \leq J_s(0) = \frac{1}{2}||Cg||_V^2$$

which yields automatically an obvious bound on the optimal control and the corresponding state.

The problem of turnpike consists in analysing whether, when the time horizon T is long enough, the minimiser for the time-evolution problem J^T is close to the steady-state one J_s. This proximity can be expressed in various manners. The strongest one is the exponential turnpike property in (5.10).

In order to perform a fine analysis of the behaviour of optimal pairs $(u^T(t), y^T(t))_{\{t\geq 0\}}$ it is convenient to characterise them in terms of the corresponding Optimality Systems (OS).

The unique minimiser $\bar{u} \in U$ of the functional J_s and the corresponding optimal state $\bar{y} \in X$ such that

$$A\bar{y} = B\bar{u} \tag{6.5}$$

can be characterised by the identity

$$\bar{u} = -B^*\bar{p} \tag{6.6}$$

where $\bar{p} \in X$ satisfies

$$A^*\bar{p} = C^* j_v(C\bar{y} - z), \tag{6.7}$$

j_v being the natural inclusion of V into V'.

Similarly, the optimal pairs (u^T, y^T) (that sometimes will be simply denoted by (u, y)) of the time-evolution problem are characterised by the state equation and the additional condition (6.6) together with the adjoint system

$$\begin{cases} p_{tt} + Ap = C^* j_v(Cy - g) \\ p(T) = p_t(T) = 0. \end{cases} \tag{6.8}$$

To prove the main turnpike result we need to ensure that the operator C entering in the cost functional allows to recover enough information on the state y and the operator B^* on the adjoint state. This is in agreement with our informal discussion in the previous section in which we indicated the relevance of the property of controllability for the turnpike phenomena to emerge. In fact the controllability of the state equation is equivalent to the observability of the adjoint one. But, because of the coupling between state and adjoint state in the OS and the fact that each variable plays a similar dual role in the equation satisfied by the other one, this kind of assumption in needed both in the state equation and in the adjoint.

Thus, we assume that the homogeneous system

$$\begin{cases} z_{tt} + Az = 0 \\ z(0) = z_0\,,\ \ z_t(0) = z_1\,, \end{cases} \tag{6.9}$$

is observable by means of C. More precisely, we suppose that, for $T > 0$ large enough, there exists an observability constant $c(T) > 0$ such that

$$|z_0|_X^2 + \|z_1\|_H^2 + |z(T)|_X^2 + \|z_t(T)\|_H^2 \leq c(T) \int_0^T \|Cz\|_V^2 dt, \tag{6.10}$$

holds for all solutions z of (6.9).

Similarly, for the homogeneous adjoint system

$$\begin{cases} q_{tt} + Aq = 0 \\ q(T) = q_0\,,\ \ q_t(T) = q_1\,, \end{cases} \tag{6.11}$$

we assume the observability inequality

$$||q_0||_H^2 + \|q_1\|_{X'}^2 + ||q(0)||_H^2 + \|q_t(0)\|_{X'}^2 \leq C(T) \int_0^T \|B^*q\|_H^2 dt, \tag{6.12}$$

for all solutions q of (6.11).

Remark 2. The observability inequality (6.12) is equivalent to the controllability of the state equation. As mentioned above this assumption is necessary to ensure that optimal pairs (u^T, y^T) are uniformly bounded with respect to T.

The turnpike property, however, requires also the extra assumption (6.10) because both state and adjoint-state play symmetric roles in the OS. ■

Remark 3. Note that, for wave-like equations, due to the finite velocity of propagation, (6.10) holds only for $T > 0$ large enough. But this is not an impediment for the turnpike property to hold since it refers to the behaviour of optimal pairs in long time horizons.

There are however models that enter in this abstract frame for which observability holds in arbitrarily small time. That is the case, for instance, for the plate model where A is the biharmonic operator. ■

Remark 4. The observability constant $c(T)$ in (6.10) is a decreasing function of T and, in fact, because A is selfadjoint and coercive (i. e. $\langle Ay, y\rangle_{X,X'} \sim ||y||_X^2$) and the system under consideration is conservative, $c(T)$ decays as $O(1/T)$ as $T \to \infty$.

Note also that, because of the time-reversibility of the abstract wave equation under consideration, estimate (6.10) is equivalent to

$$||z_0||_X^2 + \|z_1\|_H^2 \le c(T) \int_0^T \|Cz\|_V^2 dt. \tag{6.13}$$

Similar results apply for the dual observability inequality (6.12).

As observed in Remark 1 for (6.10) to hold in concrete examples, it is sufficient that the operator C entering in the penalisation term of the cost functional consists for instance on the trace of the solution of the wave equation to some subset $\mathcal{O}$ of Ω satisfying the GCC. ■

Remark 5. As observed by [28] the observability inequality (6.12) for the homogeneous adjoint system (6.11) is equivalent to the stabilisability of (6.1) with feedback $u = -B^* w_t$. Namely, solutions of the damped system

$$\begin{cases} w_{tt} + Aw + BB^* w_t = 0 \quad \text{in } (0, \infty) \\ w(0) = w_0 \,,\, w_t(0) = w_1 \,, \end{cases} \tag{6.14}$$

enjoy the property of exponential decay, i. e. the existence of $C > 0$ and $\lambda > 0$ such that

$$||w(t)||_X^2 + ||w_t(t)||_H^2 \le C \exp{(-\lambda t)}[||w_0||_X^2 + ||w_1||_H^2],\ \forall t > 0.$$

The same can be said for the state equation based on the fact that controllability implies stabilisation.■

Remark 6. *Observability for non-homogeneous state and adjoint state equations.* The observability inequalities above have been assumed to hold for the homogeneous state (6.9) and adjoint equations (6.11). However, in the proof of the turnpike property we will need to deal with non-homogeneous models submitted to an extra forcing term, as they appear in the OS. Thus, we need also to consider systems

$$\begin{cases} z_{tt} + Az = f \\ z(0) = z_0 \,,\ z_t(0) = z_1 \,, \end{cases} \tag{6.15}$$

and

$$\begin{cases} q_{tt} + Aq = C^* j_v Cg \\ q(T) = q_0 \,,\ q_t(T) = q_1 \,. \end{cases} \tag{6.16}$$

The observability inequalities above can be rewritten as follows for these non-homogeneous systems.

Consider first the adjoint problem (6.16) and let us take the scalar product with the solution w of the damped system (6.14). We get

$$(q_t, w)|_0^T - (q, w_t)|_0^T = \int_0^T [(Cg, Cw) - (B^* w_t, B^* q)] dt.$$

Taking into account that, as a consequence of the exponential decay in time, it holds

$$||w(T)||_X^2 + ||w_t(T)||_H^2 + \int_0^\infty [||w(t)||_X^2 + ||w_t(t)||_H^2] \leq C[||w_0||_X^2 + ||w_1||_H^2],$$

for all $T > 0$, and all solution w of (6.14), we deduce the existence of a constant $C > 0$, independent of $T > 0$, such that

$$||q(0)||_H^2 + ||q_t(0)||_{X'}^2 \leq C[||q_0||_H^2 + ||q_1||_{X'}^2 + \int_0^T [||Cg||_V^2 + ||B^* q||_U^2] dt. \tag{6.17}$$

A similar argument leads to the estimate

$$||z(T)||_X^2 + ||z_t(T)||_H^2 \leq C[||z_0||_X^2 + ||z_1||_H^2 + \int_0^T [||f||_H^2 + ||Cz||_V^2] dt \tag{6.18}$$

for the solutions of (6.15).

It is important to emphasize that the observability constants in (6.17) and (6.18) are independent of T. ■

Assumptions (6.10) and (6.12) are enough to deduce the turnpike property.

Theorem 1. *Assume that the observability inequalities (6.10) and (6.12) hold for the state and adjoint state equations.*

Then

$$\frac{1}{T} \min J^T \quad \overset{T\to\infty}{\longrightarrow} \quad \min J_s$$

and there exists $M > 0$ such that

$$\frac{1}{T} \int_0^T \left(||u^T(t) - \bar{u}||_U^2 + ||C(y^T(t) - \bar{y})||_V^2 \right) dt \leq \frac{M}{T} \to 0 \quad as\ T \to \infty. \tag{6.19}$$

Moreover, we have

$$\frac{1}{(b-a)T} \int_{aT}^{bT} y^T(t)\, dt \to \bar{y}, \qquad \frac{1}{(b-a)T} \int_{aT}^{bT} u^T(t)\, dt \to \bar{u}, \quad as\ T \to \infty. \tag{6.20}$$

for every $a, b \in [0, 1]$.

In fact the convergence of time-evolving optimal pairs towards the steady state ones is exponential, i. e. there exist $C > 0$ and $\lambda > 0$ such that

$$\|y^T(t) - \bar{y}\|_X + ||y_t^T(t)\|_H + \|u^T(t) - \bar{u}\|_U \leq C(e^{-\lambda t} + e^{-\lambda(T-t)})||(y_0 - \bar{y}, y_1)||_{X\times H} \qquad \forall t \in [0, T], \tag{6.21}$$

for all $T > 0$ large enough and all possible initial data $(y_0, y_1) \in X \times H$.

Remark 7. As we have seen above, for the turnpike problem to make sense, i. e. for the optimal pairs (u^T, y^T) to be uniformly bounded as $T \to \infty$, we need the system under consideration to be controllable, which is in fact equivalent to assumption (6.12). But for the turnpike property to be proved we also need the assumption (6.10) and this because, in the OS, both variables, state and adjoint-state, are coupled and play symmetric roles. □

Remark 8. The exponential version of the turnpike property (6.21) implies the averaged weaker ones (6.19) and (6.20), which is close to the notion of *measure turnpike property* ensuring that optimal trajectories and control are close to the state ones in the sense of Lebesgue measure (see e. g. [23]). ■

Proof. To simplify the presentation we drop the dependence on T on the notation of the optimal evolution pair $(u^T(t), y^T(t))$.

Substracting the optimality systems of the time-evolution problem satisfied by (y,p) and the steady-state one fulfilled by $(\bar{y},\bar{p})$ we get

$$\begin{cases} (y-\bar{y})_{tt} + A(y-\bar{y}) = B(u-\bar{u}) \\ (y-\bar{y})(0) = y_0 - \bar{y},\ (y-\bar{y})_t(0) = y_1 \\ u-\bar{u} = -B^*(p-\bar{p}) \\ (p-\bar{p})_{tt} + A(p-\bar{p}) = C^* j_v C(y-\bar{y}) \\ (p-\bar{p})(T) = -\bar{p},\ (p-\bar{p})_t(T) = 0, \end{cases}$$

where p and $\bar{p}$ are the adjoint states (the time-evolution and the steady one) representing the optimal controls.

We have

$$\begin{aligned} &\textstyle\int_0^T ||C(y-\bar{y})||_V^2\,dt - \int_0^T (B(u-\bar{u}), p-\bar{p})_H\,dt \\ &= \textstyle\int_0^T [||C(y-\bar{y})||_V^2 + ||u-\bar{u}||_U^2]\,dt \\ &= \langle (p-\bar{p})_t, (y-\bar{y})\rangle_{X',X}\Big|_0^T - ((y-\bar{y})_t, (p-\bar{p}))_H\Big|_0^T \\ &= -\langle p_t(0), (y_0-\bar{y})\rangle_{X',X} + ((y-\bar{y})_t(T), \bar{p})_H + (y_1, (p(0)-\bar{p})). \end{aligned}$$

Using assumptions (6.10) and (6.12) and, more particularly, the estimates in Remark 6 on the non-homogeneous systems we obtain

$$\int_0^T ||C(y-\bar{y})||_V^2\,dt + \int_0^T ||u-\bar{u}||_U^2\,dt \le C \tag{6.22}$$

which proves (6.19).

Let us now deduce the convergence in average for the optimal states.

First of all, using (6.22) and the estimates in Remark 6 we deduce that

$$(y^T(t), y_t^T(t)) \text{ are uniformly bounded with respect to } T>0 \text{ in } L^\infty(0,T; X\times H)$$

and

$$(p^T(t), p_t^T(t)) \text{ are uniformly bounded with respect to } T>0 \text{ in } L^\infty(0,T; H\times X').$$

To simplify the notation we shall often denote the optimal triples y^T, p^T, u^T simply by y, p, u.

Furthermore

$$\int_0^T A(y-\bar{y})dt = \int_0^T B(u-\bar{u})dt + [y_1 - y_t(T)]$$

the equality being valid in X'. Therefore we have

$$\|\frac{1}{T}\int_0^T A(y-\bar{y})dt\|_{X'} \le \frac{c}{T} + \frac{1}{T}\int_0^T \|B(u-\bar{u})\|_{X'}dt \le \frac{c}{T} + \frac{c}{T}\int_0^T \|u-\bar{u}\|_U dt\,.$$

Using the coercivity of A we get

$$\|\frac{1}{T}\int_0^T (y-\bar{y})dt\|_X \le \frac{c}{T} + \frac{c}{T}\int_0^T \|u-\bar{u}\|_U dt = O(1/T),$$

which gives the convergence in mean of the optimal trajectories. The same argument can be applied in any interval (aT, bT), with $0 \leq a < b \leq 1$. And, similar arguments, applied to the dual state equation, yield the same conclusion also holds for $p - \bar{p}$.

The proof of the exponential turnpike property requires of further developments that can be found in full detail in [37]. For the sake of completeness we recall here the main ideas.

Let us rewrite the OS in the time horizon $[0, T]$ with target $g = 0$ as

$$\begin{cases} y^T_{tt} + Ay^T = -BB^*q^T & t \in (0,T) \\ q^T_{tt} + Aq^T = C^* j_v C y^T & t \in (0,T) \\ y^T(0) = y_0 \,, y^T_t(0) = y_1 \,, \quad q^T(T) = 0 \,, \, q^T_t(T) = 0 \,. \end{cases}$$

This system defines a nonnegative operator $\mathcal{E}(T) \in \mathcal{L}(X \times H, X' \times H)$:

$$\mathcal{E}(T)(y_0, y_1) := (-q^T_t(0), q^T(0)),$$

since

$$\langle \mathcal{E}(T)(y_0, y_1), (y_0, y_1)\rangle = (-q^T_t(0), y_0)_H + (q^T(0), y_1)_H = J^T(u^T) = \min \, J^T_0 \,,$$

where J^T_0 is the cost functional with target $g = 0$.

The operator $\mathcal{E}(T)$ depends on T in a monotonic increasing manner. This is simply due to the fact that the time-horizon of control $[0, T]$ on which the cost functional is integrated grows with T. Furthermore, $\mathcal{E}(T)$ is uniformly bounded from above as $T \to \infty$. This is a consequence of exact controllability that guarantees that, in finite time $T > 0$, the solution y can be driven to the null state $(0, 0)$ by means of a suitable control $u \in L^2(0, T; U)$. This shows that $J^\infty_0(u) < \infty$ for some specific values of u, J^∞_0 being given as in (6.4). Accordingly, necessarily,

$$\min J^\infty_0(u) < \infty$$

and the operator $\mathcal{E}(\infty)$ is well defined.

Thanks to the monotone character and the uniform bound of $\mathcal{E}(T)$, and using local compactness properties of solutions of the abstract model under consideration, the operator $\mathcal{E}(T)$ has a limit $\hat{E}$ such that $\hat{E}(y_0, y_1) = (-\hat{q}_t(0), \hat{q}(0))$, where $\hat{q}$ is the adjoint state for the regulator problem, i.e. the problem with infinite horizon (see [32] and [45]),

$$\begin{cases} y^\infty_{tt} + Ay^\infty = -BB^*q^\infty & t \in (0,\infty) \\ q^\infty_{tt} + Aq^\infty - = C^* j_v C y^\infty & t \in (0,\infty) \\ y^\infty(0) = y_0 \,, y^\infty_t(0) = y_1 \\ (q^\infty(t), q^\infty_t(t)) \to (0,0), \ \text{as} \ t \to \infty. \end{cases}$$

which constitutes the OS of the infinite-horizon control problem for J^∞_0.

The regulator problem, which consists in the infinite horizon minimisation problem (6.4) for J^∞_0 is by now well understood (see e. g. [32], [6]). The optimal pair is given through a feedback operator solving the Riccati equation and ensures the exponential decay of the energy of the state as $t \to \infty$. The functional $\hat{E}$ plays the role of a Lyapunov functional.

It can also be shown that $\mathcal{E}(T)$ converges to $\hat{E}$ exponentially as $T \to \infty$. And with this we conclude the proof of the turnpike property when the target is $g = 0$. The general case can be handled by translating variables so that the non-trivial turnpike plays the role of the equilibrium around which the exponential turnpike phenomena occurs.

□

7 Further comments and open problems

Some of the main open problems arising in this field are the following:

1. The results in [18], [19] show that, suitable minimal-norm controls for exact controllability preserve the regularity of the initial data. This can be understood as an ellipticity property of the map from the data to be controlled to the control function. It would be natural to analyse similar questions in the turnpike setting and, in particular, whether, when the data are more smooth, the exponential proximity property of turnpike is fulfilled in stronger norms.

2. Obtention of the turnpike property for semilinear wave equations without smallness conditions on the target.

   The techniques developed in [38] would allow handling the semilinear wave equation with, say, a cubic-nonlinearity, but always under a smallness assumption on the target. More precisely, the linear state equation (1.1) could be replaced by the following semilinear one

$$\begin{cases} y_{tt} - \Delta y + \phi(y) = u\chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0) = y_0\,,\ y_t(0) = y_1, \end{cases} \tag{7.1}$$

   with a smooth nonlinearity ϕ, satisfying the classical growth and sign conditions to ensure the well-posedness of the problem in the energy space and the controllability in long intervals of time ([62]). But for the turnpike property to be proved by means of the techniques in [38] the target g would be needed to be small.

   Achieving similar results without the smallness condition in the target constitutes a challenging open problem.

3. The same issues arise in the context of first order hyperbolic conservation laws. In [17] the question of a time-dependent control is considered. But a systematic analysis of the turnpike property in this setting is a challenging topic.

4. Developing the hyperbolic analogues of the results in [1] and [47] on the turnpike property for control problems where the control enters through the coefficients of the wave equation or the shapes of the domains where waves evolve is an interesting and difficult subject. As mentioned above previous works focus mainly on the parabolic setting but are also limited to treat time-independent controls. Dealing with time-dependent ones would be much more challenging.

5. There is an extensive literature aimed to develop efficient numerical approximation methods for the controllability of wave-like equations ([63]). It would be interesting to analyse its consequences at the level of turnpike properties. Are some numerical schemes more suitable than others for approximating control problems in long time intervals?

   The results of this paper indicate the relevance of controllability and observability like properties to ensure the turnpike phenomenon: Accordingly, very likely, those numerical schemes that are well behaved from the controllability and observability viewpoint would also be more appropriate for turnpike purposes.

6. Recently the controllability of wave equations involving memory terms has been proved in [34]. But for this to be done the support of the control has to move to satisfy the so-called Moving Geometric Control Condition (MGCC). It would be interesting to investigate whether the turnpike property holds in this case as well, although, because of the necessity of the support of the control to move, it would be more natural to expect the periodic turnpike property to hold rather than the steady-state one.

7. The theory of stabilisation has also been developed for more complex models involving the wave equation. That is the case for instance for wave equations in 1-d networks (see e. g. [52]), or models of fluid-structure interaction (see e. g. [41], [61]). It would be natural to explore to which extent these models preserve the turnpike property.

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References

[1] G. Allaire, A. Münch, F. Periago, Long time behavior of a two-phase optimal design for the heat equation. SIAM J. Control. Optim. **48**, 5333–5356 (2010)

[2] B.D.O. Anderson, P.V. Kokotovic, Optimal Control Problems over Large Time Intervals. Automatica J. IFAC **23**, 355–363 (1987).

[3] Z. Artstein, A. Leizarowitz, Tracking periodic signals with the overtaking criterion, IEEE Transactions on Automatic Control 30 (1985), 1123-1126.

[4] V. Barbu, N. H. Pavel, Periodic optimal control in Hilbert space, Appl Math Optim 33 (1996), 169-188.

[5] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary.* SIAM J. Control Optim., 30(5), (1992), 1024–1065.

[6] A. Bensoussan, G. Da Prato, M. Delfour and S.K. Mitter, *Representation and Control of Infinite Dimensional Systems*, Second ed., Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 2007.

[7] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions and A. Porretta, *Long time average of Mean Field Games*, Netw. Heterog. Media, 7 (2012), pp. 279–301.

[8] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, A. Porretta, Long time average of mean field games with a nonlocal coupling, SIAM J. Control Optim., 51 (2013), 3558-3591.

[9] D. A. Carlson, A. Haurie, A. Jabrane, Existence of overtaking solutions to infinite-dimensional control problems on unbounded time intervals, SIAM J. Control Optim. 25 (1987), 1517-1541.

[10] D. A. Carlson, A. B. Haurie, A. Leizarowitz, Infinite Horizon Optimal Control, 2nd ed. New York: Springer Verlag, 1991.

[11] D.A. Carlson, A. Haurie and A. Jabrane, *Existence of overtaking solutions to infinite dimensional control problems on unbounded time intervals*, SIAM J Control Optim., 25 (1987), pp. 1517–1541.

[12] M. Choulli, Une introduction aux problèmes inverses elliptiques et paraboliques. Mathematiques & Applications 65. Springer-Verlag, Berlin (2009).

[13] T. Damm, L. Grüne, M. Stielerz, K. Worthmann, An exponential turnpike theorem for dissipative discrete time optimal control problems. SIAM J. Control. Optim. **52**, 1935–1957 (2014).

[14] R. Datko, *Uniform Asymptotic Stability of Evolutionary Processes in a Banach Space*, SIAM J. Math. Anal., 3 (1972), pp. 428–445.

[15] M.C. Delfour and O. Dello Sbarba, *Linear quadratic differential games: closed loop saddle points*, SIAM J. Control Optim., 47 (2008), pp. 3138–3166.

[16] R. Dorfman, P.A. Samuelson, R. Solow, Linear Programming and Economic Analysis, New York, McGraw-Hill, 1958.

[17] M. Ersoy, E. Feireisl and E. Zuazua, *Sensitivity analysis of* $1-d$ *steady forced scalar conservation laws*, J. Differential Equations 254 (2013) 3817-3834.

[18] S. Ervedoza, E. Zuazua *A systematic method for building smooth controls for smooth data* Discrete and Continuous Dynamical Systems, special issue in honor of D. L. Russell, 14, (2010), 1375-1401.

[19] S. Ervedoza and E. Zuazua, *On the numerical approximation of exact controls for waves*, Springer Briefs in Mathematics, 2013, XVII, 122 p.

[20] T. Faulwasser, M. Korda, C. N. Jones, D. Bonvin, On turnpike and dissipativity properties of continuous-time optimal control problems. arXiv: 1509.07315.

[21] E. Gilbert, Optimal periodic control: a general theory of necessary conditions, SIAM J. Control Optim. 15 (1977), 717-746.

[22] F. Grognard, A.R. Akhmetzhanov, O. Bernard, Periodic optimal control for biomass productivity maximization in a photobioreactor using natural light, Inria Research Report, Project-Teams Biocore, 7929 (2012).

[23] L. Grüne, Economic receding horizon control without terminal. Automatica J. IFAC **49**, 725–734 (2013).

[24] L. Grüne, Approximation Properties of Receding Horizon Optimal Control, Jahresber Dtsch Math-Ver (2016) 118:37. DOI 10.1365/s13291-016-0134-5.

[25] L. Grüne, M. Müller, On the relation between strict dissipativity and turnpike properties, Systems and Control Letters 90 (2016), 45-53.

[26] M. Gugat and G. Leugering L^∞ *-Norm minimal control of the wave equation: on the weakness of the bang-bang principle* ESAIM: Control, Optimisation and Calculus of Variations 14, 2008, 254-283,

[27] M. Gugat, E. Trélat, E. Zuazua, Optimal Neumann control for the 1D wave equation: Finite horizon, infinite horizon, boundary tracking terms and the turnpike property, Systems and Control Letters, 90 (2016), 61-70.

[28] A. Haraux, Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps, Portugaliae Mathematica 46, 3 (1989), 246-257.

[29] V. Isakov, Inverse problems for partial differential equations, 2nd ed. Applied Mathematical Sciences, 127. Springer, New York, 2006.

[30] V. Komornik , Rapid boundary stabilization of linear distributed systems, SIAM J. Cont. Optim. **35** (5) (1997), 1591613.

[31] J.-L. Lions *Optimal control of systems governed by partial differential equations*, Springer-Verlag, New York-Berlin, 1971.

[32] J.-L. Lions, Exact controllability, stabilization and perturbations for distributed systems, SIAM Review, 30 (1988), pp. 1-68.

[33] H. Lou, W. Wang, Turnpike properties of optimal relaxed control problems. Preprint.

[34] Q. Lü, X. Zhang, E. Zuazua, Null Controllability for Wave Equations with Memory, J. Math. pures et appli., to appear.

[35] L. W. McKenzie, Turnpike theorems for a generalized Leontief model, Econometrica 31 (1963), 165-180.

[36] D. Pighin, Long time behaviour of Optimal Control problems and the Turnpike Property, Master Thesis, Università degli Studi di Roma "Tor Vergata", December 2016.

[37] A. Porretta, E. Zuazua, Long time versus steady state optimal control, SIAM J. Control and Optim. 51 (2013), 4242-4273.

[38] A. Porretta, E. Zuazua, Remarks on long time versus steady state optimal control. Springer INdAM Series "Mathematical Paradigms of Climate Science", F. Ancona, P. Cannarsa, Ch. Jones, A. Portaluri, 15 (2016), 67-89.

[39] A. Rapaport, P. Cartigny, Turnpike theorems by a value function approach, ESAIM: Control Optim. Calc. Var. 10 (2004), 123-141.

[40] A. Rapaport, P. Cartigny, Competition between most rapid approach paths: necessary and sufficient conditions, J. Optim. Theory Appl. 124 (2005), no. 1, 1-27.

[41] J. Rauch, X. Zhang, E, Zuazua. Polynomial decay for a hyperbolic-parabolic coupled system. J. Math. Pures et Appl., **84** (9) (2005), 407–470.

[42] R. T. Rockafellar, Saddle points of Hamiltonian systems in convex problems of Lagrange, J. Optimization Theory Appl. 12 (1973), 367-390.

[43] D.L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions, SIAM Review, 20 (1978), pp. 639–739.

[44] P. A. Samuelson, The periodic turnpike theorem, Nonlinear Anal. 1(1) (1976), 3–13.

[45] E. Trélat, Contrôle Optimal : Théorie & Applications, Vuibert, Collection "Mathématiques Concrètes", 2005.

[46] E. Trélat, C. Zhang, E. Zuazua, Steady-state and periodic exponential turnpike property for optimal control problems in Hilbert spaces, preprint, 2016 (arXiv:1610.01912).

[47] E. Trélat, C. Zhang, E. Zuazua, Measure-turnpike property for dissipative infinite-dimensional optimal control problems, preprint, 2017 (hal-01442997).

[48] E. Trélat, C. Zhang, E. Zuazua, Optimal shape design for 2D heat equations in large time, preprint, 2017.

[49] E. Trélat, E. Zuazua, The turnpike property in finite-dimensional nonlinear optimal control, J. Differential Equations 258 (2015), 81-114.

[50] M. Tucsnak, An Approximation Method for Exact Controls of Vibrating Systems, SIAM J. Control Optim., 49(3), 1283-1305.

[51] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser, 2009.

[52] J. Valein, E. Zuaua, Stabilization of the wave equation on 1-d networks, SIAM J. Cont. Optim., **48** (4) (2009), 2771-2797.

[53] R. R. Wilde, P. V. Kokotovic, A dichotomy in linear control theory, IEEE Trans. Automat, Control, 17 (1972), 382-383.

[54] J. Zabczyk, *Mathematical Control Theory: An Introduction*, Systems and Control: Foundations and Applications. Birkhäuser Boston, Inc., Boston, MA, 1992.

[55] S Zamorano. The turnpike property for two-dimensional Navier-Stokes equations. arXiv:1601.04984, 2016.

[56] M. Zanon, L. Grüne, M. Diehl, Periodic optimal control, dissipativity and MPC, IEEE Transactions on Automatic Control, 10.1109/TAC.2016.2601881.

[57] A. J. Zaslavski, Existence and Structure of Optimal Solutions of Infinite-Dimensional Control Problems, Appl. Math. Optim., 42 (2000), pp. 291–313.

[58] A. J. Zaslavski, *Turnpike properties in the calculus of variations and optimal control.* Nonconvex Optimization and its Applications, 80. Springer, New York, 2006.

[59] A. J. Zaslavski, *Turnpike Phenomenon and Infinite Horizon Optimal Control*, in: Springer Optimization and Its Applications, vol. 99, Springer, Cham, 2014.

[60] A. J. Zaslavski, *Turnpike Theory of Continuous-Time Linear Optimal Control Problems, in: Springer Optimization and Its Applications*, vol. 104, Springer, Cham, 2015.

[61] X. Zhang and E. Zuazua, Long time behavior of a coupled heat-wave system arising in fluid-structure interaction, Archive for Rational Mechanics and Analysis, 184 (2007), pp. 49–120.

[62] E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, Communications in PDE, 15 (2) (1990), 205-235.

[63] E. Zuazua, *Propagation, observation, and control of waves approximated by finite difference methods.* SIAM Review, 47 (2), 2005, 197–243.

[64] E. Zuazua, *Controllability and Observability of Partial Differential Equations: Some results and open problems. Handbook of Differential Equations: Evolutionary Equations,* vol. 3, C. M. Dafermos and E. Feireisl eds., Elsevier Science, Amsterdam, 527–621, 2006.