Functional Analysis and Qualitative Theory of PDEs

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1 Introduction

1.1 Motivation

Most of the phenomena of nature, Physics and engineering require of Partial Differential Equations (PDE) modeling. Understanding the solutions of these PDEs and their properties is then of paramount importance. Each PDE has its own properties. The literature is abundant, and the field is endless growing. But, little by little, the Functional Analysis point of view is contributing to unify it. These lectures will be devoted to analyze from a Functional Analytical point of view some basic PDEs such as the heat, the wave and the Burgers equation. This will also allow us to develop some tools that will later be useful from a numerical analysis point of view.

1.2 Examples of Partial Differential Equations

Elliptic equations:

* *Laplace equation*

$$-\Delta_x u = f$$

$$\left[\Delta_x = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i}\right] \qquad N = 1, 2, 3,$$

* *Stokes equation*

$$\begin{cases} -\Delta_x \vec{u} & = & \vec{f} + \vec{\nabla}_x p \\ \operatorname{div}\vec{u} & = & 0 \end{cases}$$

* *Plate equation*

$$\Delta_x^2 u = f$$

* *The Lamé system* is 3-d elasticity

$$\mu\Delta_x\vec{u} - (\lambda + \mu)\vec{\nabla}_x \operatorname{div}_x \vec{u} = \vec{f}$$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad \vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

* *Elliptic equations* describe the stationary (time independent) solutions of evolution equations (hyperbolic and parabolic ones).

Parabolic Equations:

* The *heat equation*:

$$u_t - \Delta_x u = 0\,.$$

* The *Stokes equations*:

$$\begin{cases} \vec{u}_t - \nu\Delta\vec{u} = \vec{\nabla}p \\ \operatorname{div}\vec{u} = 0 \end{cases}$$

* *Navier-Stokes equations*:

$$\begin{cases} \vec{u}_t - \nu\Delta\vec{u} + \vec{u}\cdot\vec{\nabla}\vec{u} = \vec{\nabla}p \\ \operatorname{div}\vec{u} = 0 \end{cases}$$

* *Convection-diffusion equations*:

$$u_t - \Delta u + \operatorname{div}(\vec{f}(u)) = 0$$

Hyperbolic equations:

* The *wave equation*:

$$u_{tt} - \Delta u = 0$$

(arises in the context of acoustic waves, vibrations of strings, membranes).

* The *system of elasticity*:

$$\vec{u}_{tt} - \mu\Delta\vec{u} - (\lambda+\mu)\vec{\nabla}\mathrm{div}\vec{u} = 0\,.$$

(relevant in vitrations of 3-d elastic bodies).

* *Maxwell equations*

$$\begin{cases} \vec{E}_t &= \operatorname{curl}\vec{M} \\ \vec{M}_t &= -\operatorname{curl}\vec{E} \end{cases}$$

* *Hyperbolic conservation laws*:

$$u_t + \operatorname{div}(\vec{f}(u)) = 0.$$

Other closely related and significant models are also worth mentioning.

* *Schrödinger equation*:

$$iu_t + \Delta_u = 0,$$

(which arises in Quantum Mechanics and Optics).

* *Airy* and *KdV equations*:

$$u_t + u_{xxx} = 0$$

$$u_t + u_{xxx} + u_x + uu_x = 0$$

relevant in the theory of water waves and solitons.

* *Hamilton-Jacobi equations*:

$$u_t + H(\nabla_x u) = 0.$$

1.3 Basic principle and goal of these lectures

Understanding completely the dynamics of evolution processes is too difficult. However, for many relevant systems an asymptotic simplification process occurs and the large time behavior is often given by simpler equations allowing a quite explicit and complete analysis.

The most typical example is that of an evolution equation in which solutions converge, as time tends to infinity, to the solutions of the corresponding elliptic (time-independent) equation.

In these lectures we present some important examples in which this asymptotic simplification occurs and describe some mathematical techniques allowing to explore this issue.

The topic can be formulated to a large extent in an unifying Functional Analysis setting. These lectures will then also serve to revisit some fundamental concepts of Functional Analysis.

1.4 Some analytical tools

- Fourier & Harmonic Analysis.

- Functional Analysis (spaces, inequalities).

- Semiproup Theory.

- Interpolation.

- Distributions.

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1.5 Contents of these notes

The first two sections are devoted to analyze some fundamental properties of the heat and the wave equation. In particular we shall analyze their stability properties as $t \to \infty$ and their asymptotic simplification. In Chapter 4 we shall describe how the trend of these systems to stabilization can be understood within a more general theory of gradient systems, that also leads to important computational methods of descent for optimization and minimization of functionals. In particular, we shall describe how these ideas can be applied in the context of least squares.

We then address the Burgers equation which is a simplified version of the fundamental equations in Fluid Mechanics: The Euler and Navier-Stokes equations.

These models in Fluid Mechanics are relevant in a variety of contexts such as meterology, oceanography, climate, aeronautics, the human cardiovascular system, etc. Uniqueness and regularity of the 3-d Navier-Stokes equations is still one of the open Millennium problems. But for Burgers equations, thanks to the Hopf-Cole transformation, solutions can be computed explicitly and the vanishing viscosity limit be computed so to obtain entropy solutions of the inviscid hyperbolic version.

In the Burgers equation viscosity or diffusion and nonlinear convection compete. This is then a natural model to develop splitting methods, following the book by R. Glowinski [13].

The Burgers equation also motivates the last two topics addressed in the Notes. On one hand, elliptic equations of convection-diffusion nature in which existence of solutions can be proved combining the variational theory of elliptic equations and fixed point arguments. On the other, the theory of nonlinear semigroups, that based on time.discretization, allows solving a large class of nonlinear diffusion problem, including Burgers like equations.

2 The Heat Equation

2.1 The Dirichlet Problem

Given a bounded domain Ω we consider the heat equation with Dirichlet boundary conditions:

$$\begin{cases} u_t - \Delta u = 0 & \Omega \times (0, \infty) \\ u = 0 & \partial\Omega \times (0, \infty) \\ u(x, 0) = \varphi(x) & \Omega. \end{cases}$$

Here $u = u(x,t)$ where $x = (x_1, \dots, x_n) \in \mathbb{R}$, $t > 0$ and $\Delta = \Delta_x = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ and $\cdot_t = \frac{\partial \cdot}{\partial t}$.

Fourier Analysis allows obtaining the explicit form of solutions.

Consider first the eigenvalue problem

$$\begin{cases} -\Delta\phi_j = \lambda_j\phi_j & \text{in } \Omega \\ \phi_j = 0 & \text{on } \partial\Omega, j = 1, \dots, \infty \end{cases}$$

The spectral theory for compact self-adjoint operators in Hilbert spaces allows showing that the problem admits an increasing sequence of positive eigenvalues, of finite multilplicity, tending to infinity

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \ldots \to \infty$$

so that the corresponding eigenfunctions $\{\phi_j\}$ constitute an orthonormal basis of $L^2(\Omega)$.

Solutions of the heat equation can now be easily developed in Fourier series in this eigenfunctions basis:

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k \, e^{-\lambda_k t} \, \phi_k(x)$$

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k \, \phi_k(x)$$

$$\varphi_k = \int_\Omega \varphi(x) \, \phi_k(x) dx.$$

According to the orthogonality property of eigenfunctions the following holds for the time evolution of the L^2-norm of solutions:

$$||u(t)||^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \varphi_k^2 \, e^{-2\lambda_k t}.$$

This identity illustrates the well-posedness of the system in the forward sense[*] [†] and provides an easy bound for the semigroup map $S(t)$ that associates to any initial datum φ the solution $u(t)$ at time t:

$$||S(t)||_{\mathcal{L}(L^2(\Omega))} = e^{-\lambda_1 t}.$$

[*]Backward uniqueness for the heat equation is also an important topic in the context of inverse and control problems theory. However, the backward heat equation is one of the most paradigmatic example of ill-posed problem in the sense of Hadamard. However, energy estimates allow proving that, indeed, backward uniqueness holds and actually to yield an upper bound of the growth rate of the energy of solutions in time in terms of the frequency number of the initial datum.

[†]This Fourier series representation also allows proving, in agreement with the time-analyticity of solutions (since, indeed, the heat equation is one of the so-called analytic semigroups) that the vanishing properties of solutions in a finite time interval necessarily extend to infinity time. And this is often of much use in control theory as well. For instance, if $u(x_0, t) \equiv 0$ for $t \in [0, T]$, then, necessarily, $u(x_0, t) \equiv 0$ for al $t > 0$. And, it is not hard to see that this holds if and only if either x_0 is in the zero set of some eignefunction ϕ_k or if $u \equiv 0$.

By the contrary, the same identity illustrates the strong *time-irreversibility* of the system:

$$||u(0)||^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \varphi_k^2 = \sum_{k=1}^{\infty} u_k^2(t) e^{2\lambda_k t}.$$

Time irreversibility and the *smoothing property* of the heat semigroup are strongly linked. The following holds:

$$\varphi = u(0) \in L^2(\Omega) \Rightarrow u(t) \in H^s(\Omega), \ \forall\, s > 0.$$

Indeed,

$$||u(t)||^2_{H^s(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^{2s}\, u_k^2(t) = \sum_{k=1}^{\infty} \lambda_k^{2s}\, e^{-2\lambda_k t}\, \varphi_k^2 \leq C_s \sum_{k=1}^{\infty} \varphi_k^2 < \infty.$$

Consequently, the heat equation posesses the following main properties:

- Smoothing
- Time Irreversibility
- Dissipation of energy.

The energy dissipation law can also be obtained easily by the *energy method* that consists, roughly, on multiplying the equation by suitable functions of the unknown and integrating by parts. Indeed, integrating with respect to the space variable x in the identity

$$(u_t - \Delta u)u = 0$$

it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2\, dx = 0.$$

In other words,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 = -\int_\Omega |\nabla u|^2\, dx$$

and, using the Poincaré inequality, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \leq -\lambda_1 \int_\Omega u^2$$

and consequently,

$$||u(t)||_{L^2(\Omega)} \leq e^{-\lambda_1 t}||\varphi||_{L^2(\Omega)}.$$

According to this computation solutions of the heat equation on a bounded domain Ω decay exponentially in time with a rate $\lambda_1(\Omega)$. This is in agreement with the prediction we did by using the Fourier expansion of solutions. It is however important to note that the energy method is more flexible and does not require an orthonormal basis of eigenfucntions to be applied.

The computation above is based on the following characterization of the best constant in the Poincaré inequality:

$$\lambda_1(\Omega) = \min_{\varphi \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla \varphi|^2 \, dx}{\int_\Omega \varphi^2 \, dx}$$

which does indeed yield

$$\int_\Omega |\nabla \varphi|^2 \geq \lambda_1 \int_\Omega \varphi^2.$$

In order to better understand the asymptotic behavior of solutions as a function of the domain Ω where the equation holds it is important to take into account that

$$\lambda_1(\Omega) \text{ decreases as } \Omega \text{ increases.}$$

More precisely, by scaling the domain Ω by means of a constant factor $R > 0$

$$\Omega \longrightarrow R\Omega$$

we see that

$$\lambda_1(\Omega) \rightarrow \lambda_1(R\Omega) = \frac{1}{R^2}\lambda_1(\Omega).$$

In particular, the computations above do not yield any decay rate for the *Cauchy problem* since, as $R \to \infty$, $\lambda_1(R\Omega) \to 0$. Thus, when $\Omega = \mathbb{R}^n$, which is a mathematical idealization of a very large domain in which the boundary effects do not have a significant influence on the solution, the computation above does not provide any decay rate..

The following question arises: *Do solutions decay at all when* $\Omega = \mathbb{R}^n$*?*

2.2 The Cauchy Problem

Consider now the Cauchy problem in the whole space:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x,0) = \varphi(x) & \text{in } \mathbb{R}^N. \end{cases}$$

The solution of the Cauchy problem can be explicitly written by *convolution with the gaussian heat kernel*:

$$\boxed{u = G(\cdot, t) \underset{x}{*} \varphi(\cdot)}.$$

More explicitly,

$$u(x,t) \;=\; \int_{\mathbb{R}^N} G(x-y,t)\,\varphi(y)\,dy$$

where

$$G(x,t) \;=\; (4\pi t)^2 \exp\left(\frac{-|x|^2}{4t}\right)$$

is the fundamental solution of the heat equation, the gaussian kernel.

It satisfies the following properties:

- $G_t \,-\, \Delta G = 0, \quad$ in $\mathbb{R}^N \,\times\, (0,\infty)$
- $G(t) \;\to\; \delta_0 \;=$ Dirac delta at $x = 0$, as $t \to 0^+$;
- Conservation of mass:
$$\int G(x,t)dx \;=\; 1, \quad \forall t > 0.$$
- L^p-decay
$$||G(\cdot,t)||_{L^p(\mathbb{R}^N)} \;=\; C_p t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}.$$
  In particular
$$||G(t)||_{L^1} \;=\; \int G\,dx \;=\; 1, \quad ||G(t)||_{L^\infty} \;\le\; ct^{-\frac{N}{2}}.$$
- L^p-decay of derivatives.
$$||D_x^\alpha G(\cdot,t)||_{L^p(\mathbb{R}^N)} \;=\; C_{p\alpha} t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{2}}$$
- We also observe that that the fundamental solution has the *self-similar form*:
$$G \;=\; t^{-\frac{N}{2}}\, f\left(\frac{x}{\sqrt{t}}\right), \quad f(z) \;=\; (4\pi)^{-\frac{N}{2}} \exp\left(-\frac{z^2}{4}\right).$$

The fundamental solution of the heat equation can be computed explicitly in several forms. One of them is by direct application of the basic properties of the *Fourier transform.* The fundamental solution solves the following system in the physical space:

$$\begin{cases} u_t \,-\, \Delta u \;=\; 0 \\ u(0) \;=\; \delta_0. \end{cases}$$

Applying Fourier transform in space we get:

$$\begin{cases} \widehat{u}_t \,+\, |\xi|^2 \widehat{u} \;=\; 0 \\ \widehat{u}(0) \;\equiv\; 1. \end{cases}$$

The solution in the Fourier variables can then be obtained explicitly:

$$\widehat{u}(\xi, t) \;=\; e^{-|\xi|^2 t}.$$

Finally, applying the inverse Fourier transform we easily obtain that

$$u(x,t) \;= G(x,t) \;=\; (4\pi t)^{-N/2} \exp\left(\frac{-|x|^2}{4t}\right).$$

But, the Fourier analysis provides more information than the explicit form of the solution. Indeed, it indicates that the heat equation dissipates exponentially each Fourier component of the solution with a rate $|\xi|^2$. Consequently,

→ The dissipation rate is very high for the high frequencies.

→ It is very slow for the low ones.

Of course, there is no unifom exponential decay rate. However, one can get a compromise between the strong decay of high frequencies and the weak decay rate of the low ones by introducing extra assumptions on the integrability of the initial datum.

It is important to note that, if no additional assumption on the initial datum is imposed one can not obtain any decay rate. More precisely, the norm of the semigroup $S(t)$ associated with the Cauchy problem as a bounded linear operator form $L^2(\mathbb{R}^N)$ into itself is one for all $t > 0$.

This fact corresponds to a general result in the context of dissipative semigroups in Banach spaces

$$S(t) \;=\; e^{At} \;:\; X \to X$$

guaranteeing that either:

$$||S(t)||_{\mathcal{L}(X,X)} \;=\; 1, \quad \forall t > 0$$

or

$$||S(t)||_{\mathcal{L}(H,H)} \;\leq\; C\, e^{-\omega t} \qquad \forall t > 0$$

for suitable constants $C, \omega \;>\; 0$.

In particular, if we have a polynomial decay rate of the form

$$||S(t)||_{\mathcal{L}(X,X)} \;\leq\; c\, t^{-\sigma},$$

then, necesarilly, the semigroup decays exponentially as well.

In the particular case of the semigroup associated with the Cauchy problem for the heat equation, the semigroup is of unit norm. This fact is closely related to the fact that the gaussian heat kernel $G(t)$ is of unit norm in $L^1(\mathbb{R}^N)$ for all $t > 0$ as well. Indeed, according to Youngs's inequality we have

$$||S(t)\varphi||_{L^2(\mathbb{R}^N)} = ||G(t) * \varphi||_{L^2(\mathbb{R}^N)} \leq ||G(t)||_{L^1(\mathbb{R}^N)} ||\varphi||_{L^2(\mathbb{R}^N)} = ||\varphi||_{L^2(\mathbb{R}^N)}$$

and this bound, which does not provide any decay rate, is sharp.

We have discussed the main properties of the gaussian heat kernel. But, in fact, all solutions of the heat equation have similar properties and they can be obtained by the energy method, i.e. multiplying the equation by functions of the unknown and integrating by parts:

- *Conservation of mass.*
  Integrating the heat equation with respect to x we get:

$$0 = \int_{\mathbb{R}^N} u_t \, dx - \int_{\mathbb{R}^N} \Delta_x u \, dx = \frac{d}{dt} \int u \, dx.$$

  Consequently,

$$\int_{\mathbb{R}^N} u(x,t) \, dx = \int_{\mathbb{R}^N} \varphi(x) \, dx, \qquad \text{for all} \quad t > 0.$$

- *Energy dissipation law.*
  Multiplying by u and integrating w.r.t. x we get:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} u^2 \, dx = - \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.$$

- L^p-decay.
  Applying Young's inequality in the representation formula of the solution as convolution of the heat kernel with the initial datum we obtain:

$$\begin{aligned} ||u(t)||_{L^p(\mathbb{R}^N)} &= ||G(t) * \varphi||_{L^p(\mathbb{R}^N)} \leq ||G((t)||_{L^p(\mathbb{R}^N)} ||\varphi||_{L^1(\mathbb{R}^N)} \\ &\leq C_p ||\varphi||_{L^1(\mathbb{R}^N)} t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}. \end{aligned}$$

  This is true for all $p \geq 1$. The maximal decay rate is achieved for $p = \infty$, the decay rate being $N/2$. When $p = 1$ the L^1 norm does not decay. This is in agreement with the property of conservation of mass.

  Note however that the L^p decay property is guaranteed by assuming that the initial datum is in $L^1(\mathbb{R}^N)$. The semigroup does not decay as a bounded linear operator from $L^p(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$.

- *Comparison of solutions*: Using the positivity of the heat kernel G it can also be easily seen that $\varphi \geq \widehat{\varphi}$ implies that the associated solutions are ordered as well: $u \geq \widehat{u}$.

This property may also be obtained by mutiplying the equation by $\mathrm{sgn}^-(u - \widehat{u})$. Indeed,

$$0 = \int_{\mathbb{R}^N} (u_t - \widehat{u}_t)\mathrm{sgn}^-(u - \widehat{u}) - \underbrace{\int_{\mathbb{R}^N} \Delta(u - \widehat{u})\,\mathrm{sgn}^-(u - \widehat{u})\,dx}$$

$$\geq 0$$

$$\geq \frac{d}{dt}\int_{\mathbb{R}^N} \left|[u - \widehat{u}]^-\right|\,dx.$$

Here sgn^- stands for the function taking value -1 for $s \leq 0$ and value 0 for $s \geq 0$. On the other hand, $[s]^-$ is the negative part function taking value s for $s \leq 0$ and 0 for $s \geq 0$.

Consequently, taking into account that $\varphi \geq \widehat{\varphi}$ we have

$$\int_{\mathbb{R}^N} |[\varphi - \widehat{\varphi}]^-|dx = 0$$

and we deduce immediately that

$$\int_{\mathbb{R}^N} |[u - \widehat{u}]^-|\,dx = 0$$

for all $t > 0$ which is equivalent to the fact that $u \geq \widehat{u}$.

In order to justify this computation and to avoid the technical difficulties related with the lack of smoothness of the sgn^- function one has to use a regular approximation β_ε of the sgn^- function, such that β_ε is of class C^1, vanishes at the origin, and is non-decreasing. Using the Green formula one then gets

$$-\int_{\mathbb{R}^N} \Delta w\,\beta_\varepsilon(w)dx = \int_{\mathbb{R}^N} \nabla w \cdot \beta'_\varepsilon(w)\nabla w = \int_{\mathbb{R}^N} \beta'_\varepsilon(w)|\nabla w|^2 \geq 0.$$

In this way we conclude the decreasing character of the integral

$$\int_{\mathbb{R}^N} B_\varepsilon(u - \widehat{u})dx$$

where $B_\epsilon(s) = \int_0^s \beta_\varepsilon(z)dz$. Passing to the limit as ε tends to zero we deduce that the integral $\int_{\mathbb{R}^N} |[u - \widehat{u}]^-|\,dx$ decreases as well.

We have seen that most of the properties of solutions of the heat equation can be obtained in two different ways: Using the explicit expression of solutions by convolution with the keat kernel or by integration by parts. The proofs based on integration by parts are much more robust than those that use the explicit representation formula of solutions.

The following is an interesting alternative proof of the L^p-decay property. This method applies to nonlinear parabolic equations and is also useful to obtain L^p estimates for solutions of semilinear elliptic equations. As far as we know, the application of this technique in the context of parabolic equations is due to L. Véron [29].

- Step 1: Multiply the equation by $|u|^{p-2}u$. We get

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N} (p-1)\ |\nabla u|^2\ |u|^{p-2} dx = 0 \\ \Rightarrow\ \frac{d}{dt}\int_{\mathbb{R}^N} |u|^p dx \ &\leqslant\ 0 \\ \Rightarrow\ \|u(t)\|_{L^p(\mathbb{R}^N)} \ &\leqslant\ \|u_0\|_{L^p(\mathbb{R}^N)},\quad \forall t > 0. \end{aligned}$$

- Step 2: Use of Sobolev's inequality. For instance, in three space dimensions ($N = 3$) the Sobolev inequality reads.

$$\left(\int_{\mathbb{R}^N} |u|^6\right)^{1/6} \leqslant c \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{1/2}.$$

On the other hand, from the identities of the first step we have

$$\frac{1}{p}\frac{d}{dt}\int |u|^p dx + (p-1)\int |\nabla u|^2\ |u|^{p-2} dx = 0,$$

and, taking into account that

$$\int |\nabla u|^2\ |u|^{p-2} dx = \frac{4}{p^2}\int |\nabla[(u)^{p/2}]|^2 dx$$

we get

$$\frac{1}{p}\frac{d}{dt}\int |u|^p dx + \frac{4(p-1)}{p^2}\int |\nabla (u)^{p/2}|^2 dx = 0.$$

The application of the Sobolev inequality in this case yields:

$$\int |\nabla (u)^{p/2}|^2 dx \geqslant c\left(\int |u|^{3p}\right)^{1/3}$$

and therefore

$$\frac{1}{p}\frac{d}{dt}\int |u|^p dx + \frac{c(p-1)}{p^2}\left(\int |u|^{3p}\right)^{1/3} \leqslant 0.$$

- Step 3: We now use classical interpolation inequalities (which are in fact a consequence of Hölder's inequality):

$$\|u\|_{L^p} \leqslant \|u\|_{L^1}^{2p/[3p-1]} \|u\|_{L^{3p}}^{(p-1)/[3p-1]}.$$

Then, using the fact that the L^1-norm decreases,

$$\frac{1}{p}\frac{d}{dt}\int |u|^p dx + \frac{c(p-1)}{p^2}\left[\frac{\|u\|_{L^p}}{\|u\|_{L^1}^{2p/(3p-1)}}\right]^{p(3p-1)/3(p-1)} \leqslant 0$$

$$\Rightarrow \frac{d}{dt}\int |u|^p dx + \frac{c(p-1)}{p^2\|\varphi\|_{L^1}^{2p^2/(p-1)}}\left(\int |u|^p dx\right)^{(3p-1)/3(p-1)} \leqslant 0.$$

Consequently, the function

$$\phi(t) = \int_{\mathbb{R}^N} |u|^p dx$$

satisfies

$$\frac{d\phi}{dt} + C_p(\|\varphi\|_{L^1})\phi^{(3p-1)/3(p-1)} \leq 0.$$

Solving this differential ineguality we get:

$$\phi(t) \leq C_p t^{-\frac{3}{2}(p-1)} \Rightarrow \|u(t)\|_{L^p} \leq C_p\|\varphi\|_{L^1} t^{-\frac{3}{2}(1-\frac{1}{p})}.$$

Note that this corresponds, when $N = 3$, to the estimate we got using Young's ineguality in the convolution identity.

This method of proving L^p estimates applies to more general nonlinear evolution equations like, for instance,

$$u_t - \Delta u + \operatorname{div}(\vec{f}(u)) = 0\,.$$

Indeed,

$$\int_{\mathbb{R}^N} \operatorname{div}(\vec{f}(u))|u|^{p-2}u dx = -\int_{\mathbb{R}^N} (p-1)\vec{f}(u)|u|^{p-2}\nabla u = -\int_{\mathbb{R}^N} \operatorname{div}(\vec{F}_p(u))dx = 0,$$

where

$$\vec{F}_p(z) = \int_0^z \vec{f}(s)|s|^{p-2}(p-1)ds$$

and therefore all the contributions coming from the nonlinear term cancel in this computation, leading to the same result as for the linear heat equation.

2.3 Development of functions in the Dirac basis. Large time behavior

According to the estimates above, when the initial datum φ of the Cauchy problem belongs to $L^1(\mathbb{R}^N)$, $t^{\frac{N}{2}(1-\frac{1}{p})}u(t)$ is a bounded trajectory in $L^p(\mathbb{R}^N)$ is $t \to \infty$.

The following question then arises naturally: *Does the limit*

$$\lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{p})}u(t)$$

exist in $L^p(\mathbb{R}^N)$*, and, if yes, can we compute it explicitly.*

There are, at least, three methods to answer to these questions:

- Developing functions in the basis of delta functions.
- Self-similar variables.
- Scaling.

Let us discuss the first method, based on the possibility of developing functions on the basis constituted by the Dirac delta and its derivatives, the coefficients being the momenta of the function to be developed.

The following holds:

Decomposition Lemma. ([9]). *Assume that* $f \in L^1(\mathbb{R}^N; 1+|x|)$*, i.e.*

$$\int_{\mathbb{R}^N} |f(x)|(1+|x|)dx < \infty .$$

Then, there exists $\vec{F} \in (L^1(\mathbb{R}^N))^N$ *such that*

$$f \;=\; \int_{\mathbb{R}^N} f(x)dx\delta_0 + div(\vec{F})$$

and

$$\|\vec{F}\|_{L^1(\mathbb{R}^N)} \;\leq\; C_N\| \, |x|f \, \|_{L^1(\mathbb{R}^N)}.$$

Moreover, if

$$f \in L^1(\mathbb{R}^N; 1+|x|^2)$$

then,

$$f = \int f(x)dx\delta_0 - \int f(x)xdx \cdot \nabla\delta_0 + \sum_{|\alpha|=2} D^\alpha F_\alpha$$

with

$$\sum_{|\alpha|=2} \|F_\alpha\|_{L^1} \leq C_N\| \, |x|^2 f\|_{L^1} .$$

This decomposition formula can be easily used to analyze the asymptotic behavior of solutions. Recall that

$$u(x,t) = \left[G(\cdot,t) * \varphi(\cdot)\right](x).$$

On the other hand, according to the decomposition formula,

$$\varphi = \int_{\mathbb{R}^N} \varphi dx \delta_0 + \operatorname{div}(\vec{\phi})$$

and, consequently,

$$\begin{aligned} u = G * \left[\int_{\mathbb{R}^N} \varphi dx\, \delta_0\right] + G * \operatorname{div}(\vec{\phi}) = \int_{\mathbb{R}^N} \varphi dx \left[G * \delta_0\right] + \nabla G * \vec{\phi} \\ = \int_{\mathbb{R}^N} \varphi dx\, G + \nabla G * \vec{\phi}. \end{aligned}$$

In other words,

$$u \;-\; \int_{\mathbb{R}^N} \varphi dx\, G \;=\; \nabla G * \vec{\phi}$$

and therefore

$$\|u - \int_{\mathbb{R}^N} \varphi dx G\|_{L^p} \;\leq\; \|\nabla G\|_{L^p} \|\vec{\phi}\|_{L^1} \leq C\|\nabla G\|_{L^p}\|x\varphi\|_{L^1}.$$

But, taking into account that,

$$G = t^{-N/2} f(x/\sqrt{t}) \;\Rightarrow\; \nabla_x G = t^{-(N+1)/2} F(x/\sqrt{t})$$

we have

$$\|\nabla_x G\|_{L^p} \;\leq\; C_p t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{1}{2}}$$

and therefore

$$t^{\frac{N}{2}(1-\frac{1}{p})} \|u - \int_{\mathbb{R}^N} \varphi dx\, G\|_{L^p} \;\leq\; C_p t^{-1/2}\,.$$

We have proved the following result:

Theorem.

- *If* $\varphi \in L^1(\mathbb{R}^N; 1 + |x|)$, *then*

$$\|u - \int_{\mathbb{R}^N} \varphi dx\, G\|_{L^p} t^{\frac{N}{2}(1-\frac{1}{p})} \;\leq\; C_p t^{-1/2}\,.$$

- *If* $\varphi \in L^1(\mathbb{R}^N)$, *then*

$$t^{\frac{N}{2}(1-\frac{1}{p})}\|u - \int_{\mathbb{R}^N} \varphi dx\, G\|_{L^p} \to 0\,,\ t \to \infty\,.$$

Summarizing, roughly speaking, we can say that

$$u \ \sim \ \int_{\mathbb{R}^N} \varphi dx\, G\,, \quad \text{as} \quad t \to \infty\,.$$

Proof.

- The first statement has been proved before.
- The second result can be proved by density. Indeed,

$$\exists u_{0,\varepsilon} \in C_0^\infty(\mathbb{R}^N) : u_{0,\varepsilon} \xrightarrow[\epsilon \to 0]{L^1} u_0; \int u_{0,\varepsilon} = \int u_0.$$

Using this approximating sequence and applying the previous result we have:

$$\|u - \int u_0\, dx\, G\|_{L^p}\, t^{\frac{N}{2}(1-\frac{1}{p})} \leqslant \|u_\varepsilon - \int u_{0,\varepsilon}\, dx\, G\|_{L^p}\, t^{\frac{N}{2}(1-\frac{1}{p})} + \|u_\varepsilon - u\|_{L^p}\, t^{\frac{N}{2}(1-\frac{1}{p})}$$

$$\leqslant C_\varepsilon t^{-1/2} + C\|u_0 - u_{0,\varepsilon}\|_{L^1} \leqslant \frac{\delta}{2} + \frac{\delta}{2}.$$

The latter can be guaranteed by taking ε small enough (depending on δ), and then, the first one, once ε is fixed, taking t large enough.

Using more terms of the development on the Dirac basis we can obtain more terms on the asymptotic development of the solution as $t \to \infty$ as well. In this way we can show, as $t \to \infty$,

$$u \sim \int_{\mathbb{R}^N} u_0 dx\, G - \int_{\mathbb{R}^N} x u_0 dx \cdot \nabla G + \ldots$$

It is important to observe that every time we add a term in the asymptotic expansion (containing the higher order momentum of the initial datum and the corresponding derivative of the Gaussian kernel) we obtain an extra decay rate of the order of $t^{-\frac{1}{2}}$.

Proof of the Decomposition Lemma. We have to show that

$$\boxed{u_0 = \int u_0 dx \delta_0 + \operatorname{div}(\vec{V_0}).}$$

In one space dimension ($N = 1$) we can compute V_0 explicitly as follows:

$$V_0(x) = \begin{cases} -\int_x^\infty u_0\, dx, & x > 0 \\ \int_{-\infty}^x u_0\, dx, & x < 0. \end{cases}$$

For this function V_0 we have

$$\frac{dV_0}{dx} = u_0 \quad \text{for } x > 0,\, x < 0$$

and

$$[V_0]_{x=0} = -\int u_0\, dx.$$

Consequently,

$$\frac{dV_0}{dx} = u_0 - \int u_0\, dx\, \delta_0.$$

In several space dimensions the same can be proved integrating along rays. We then obtain

$$\vec{V_0} = -x \int_1^\infty t^{N-1}\, u_0(tx) dt.$$

■

The same formula allows to get a L^p-version of the decomposition result. Applying Minkowski's inequality we get

$$\|\vec{V_0}\|_{L^p_x} \leqslant \int_1^\infty t^{N-1} \|xu_0(tx)\|_{L^p_x} dt.$$

Taking into account that

$$\|xu_0(tx)\|_{L_p} = t^{-1-N/p}\|xu_0(x)\|_{L^p_x} \leqslant \|xu_0(x)\|_{L^p_x} \int_1^\infty t^{N(1-\frac{1}{p})-2} dt$$

we obtain

$$\|\vec{V_0}\|_{L^p_x} \leqslant \|xu_0(x)\|_{L^p_x} \int_1^\infty t^{N(1-\frac{1}{p})-2} dt.$$

The last integral converges if and only if

$$N(1-\frac{1}{p}) - 2 < -1 \Leftrightarrow N(\frac{p-1}{p}) - 2 < 1 \Leftrightarrow p < N/(N-1).$$

This yields the following result:

Theorem 2.1. *([9])*

- *Assume that* $1 \leq p < \frac{N}{N-1}$, *and* $f \in L^1(\mathbb{R}^N)$, $|x|f \in L^p(\mathbb{R}^N)$ *then there exists* $\vec{F} \in (L^p(\mathbb{R}^N))^N$ *such that*

$$f = \int_{\mathbb{R}^N} f(x) dx \delta_0 + div\vec{F}.$$

- *If* $\frac{N}{N-1} < p \le \infty$, *under the assumption that* $|x|f \in L^p(\mathbb{R}^N)$ *there exists* $\vec{F} \in L^p(\mathbb{R}^n)$ *such that*
$$f = div(\vec{F}).$$

Proof. We indicate the main steps of the proof in the first case. The second one is left as an exercise.

- **Case 1.** $1 \le p < \frac{N}{N-1}$. We set
$$F_j = -\int_0^1 x_j f\left(\frac{x}{t}\right) \frac{1}{t^{N+1}} dt.$$

Then,

$$\|F_j\|_p \le \int_0^1 \|x_j f\left(\frac{x}{t}\right)\|_p \frac{1}{t^{N+1}}\, dt = \int_0^1 t^{1+\frac{N}{p}} \|xf\|_p \frac{1}{t^{N+1}}\, dt$$
$$= \|xf\|_p \int_0^1 t^{N(\frac{1}{p}-1)}\, dt \le C(p,N)\|xf\|_p$$

with $C(p,N) = \frac{p}{|N-(N-1)p|} < \infty \iff \frac{1}{p} - 1 > -\frac{1}{N} \iff p < \frac{N}{N-1}$.

The definition of the remainder term F_j can be motivated as follows:

$$\langle f - \int_{\mathbb{R}^N} f\, dx \delta_0, \varphi\rangle = \int_{\mathbb{R}^N} f(x)\left[\varphi(x) - \varphi(0)\right] dx$$
$$= \int_{\mathbb{R}^N} f(x) \int_0^1 x \cdot \nabla_x \varphi(tx)\, dt\, dx = -\int_{\mathbb{R}^N} \varphi(x) \mathrm{div} \underbrace{\left[x \int_0^1 f\left(\frac{x}{t}\right) \frac{dx}{t^{N+1}}\right]}_{F}.$$

■

Higher order generalizations can also be easily obtained. In particular, it follows that

$$f = \sum_{|\alpha| \le k} \frac{(-1)^{|\alpha|}}{\alpha!} \int_{\mathbb{R}^N} f(x) x^\alpha\, dx D^\alpha \delta_0 + \sum_{|\alpha|=k+1} D^\alpha F_\alpha,$$

with F_α belonging to L^1, under the assumption that $f \in L^1\left(1 + |x|^{k+1}\right)$.

Applying this decomposition results one can get the asymptotic expansion of solutions as $t \to \infty$.

Consider the Cauchy problem
$$u_t - \Delta u = 0$$
with initial datum $\varphi = \varphi(x)$.

We have, $u = G * \varphi$ with, as indicated above, $G = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$. We know that

$$G * D^\alpha \delta_0 = D^\alpha \left[G * \delta_0\right] = D^\alpha G.$$

Applying then the decomposition formula

$$\varphi = \sum_{|\alpha| \le k} \frac{(-1)^{|\alpha|}}{|\alpha|!} \int_{\mathbb{R}^N} \varphi(x) x^\alpha \, dx D^\alpha \delta_0 \quad + \sum_{|\alpha|=k+1} D^\alpha F_\alpha \, .$$

we obtain

$$u = \sum_{|x| \le k} \frac{(-1)^{|\alpha|}}{|\alpha|!} \int_{\mathbb{R}^N} \varphi(x) x^\alpha \, dx D^\alpha G \quad + \underbrace{\sum_{|\alpha|=k+1} F_\alpha * D^\alpha G}_{R}$$

with

$$\|R\|_{L^p(\mathbb{R}^N)} \quad \le \quad C_p \| |x|^{k+1} \varphi \|_{L^1(\mathbb{R}^N)} t^{-\frac{N}{2}(1-\frac{1}{p}) - \frac{|\alpha|}{2}}.$$

The later estimate is sharp.

The decomposition results we have proved on the basis of the Dirac delta and its derivatives are closely related with the Taylor power series expansion of the Fourier transform of the function to be developed. Indeed, the values of the Fourier transform and its derivatives at the origin can be easily computed

$$\begin{aligned} \widehat{f}(\xi) &= \int_{\mathbb{R}^N} f(x) e^{-2\pi i x \cdot \xi} \, dx \\ \widehat{f}(0) &= \int_{\mathbb{R}^N} f(x) \, dx \\ D\widehat{f}(0) &= -2\pi i \int_{\mathbb{R}^N} x f(x) \, dx \\ &\cdots \end{aligned}$$

Therefore, provided $\widehat{f}$ is real analytic we have

$$\widehat{f}(\xi) = \sum_{k=0}^{\infty} \frac{D^k \widehat{f}(0)}{k!} \xi^k.$$

We get

$$f(x) = \sum_{k=0}^{\infty} \frac{D^k \widehat{f}(0)}{k!} \mathcal{F}^{-1}\left(\xi^k\right)$$

which, taking into account that $\widehat{\delta_0} = 1$, i.e. $\delta_0 = \mathcal{F}^{-1}(1)$, and the resulting expressions for $\mathcal{F}^{-1}(\xi^k)$, yields the infinite order asymptotic expansion we have obtained rigorously at finite order in the previous results.

The decomposition formulas obtained above are also closely related with *Hardy inequalities*. Indeed, observe that, for instance, the decomposition formula

$$f \;=\; \int_{\mathbb{R}^N} f\,dx\delta_0 - \operatorname{div}(\vec{F})$$

(which holds, according to the previous results, when $f \in L^1$, $|x|f \in L^p$, $1 \le p < \frac{N}{N-1}$ with $\vec{F} \in (L^p)^N$) yields

$$\left|\int_{\mathbb{R}^N} |x|f(x)\frac{[\varphi(x)-\varphi(0)]}{|x|}dx\right| = \left|-\int_{\mathbb{R}^N} -\vec{F}\cdot\nabla\varphi\right| \le \|\vec{F}\|_{L^p}\|\nabla\varphi\|_{L^{p'}} \le C\|\,|x|f\|_{L^p}\|\nabla\varphi\|_{L^{p'}}$$

if and only if $1 \le p < N/(N-1)$. By duality this yields the Hardy inequality,

$$\|\frac{\varphi(x)-\varphi(0)}{|x|}\|_{L^{p'}(\mathbb{R}^N)} \le C_{p'}\|\nabla\varphi\|_{L^{p'}(\mathbb{R}^N)} \qquad \forall\,\varphi \in C_0^\infty(\mathbb{R}^N) \qquad N < p' \le \infty.$$

The reverse is also true the extremal case $p = 1$ being excluded.

By the contrary, the decompostion formula

$$f = \operatorname{div}(\vec{F}), \quad p > N/(N-1)$$

yields

$$\left|\int_{\mathbb{R}^N} |x|f(x)\frac{\varphi(x)}{|x|}dx\right| = \left|-\int_{\mathbb{R}^N} \vec{F},\nabla\varphi\,dx\right| \leqslant\; \|\vec{F}\|_{L^p}\;\|\nabla\varphi\|_{L^{p'}} \leqslant\; c_p\|\,|x|\,f\|_{L^p}\;\|\nabla\varphi\|_{L^{p'}}$$

and, consequently,

$$\|\frac{\varphi(x)}{|x|}\|_{L^{p'}(\mathbb{R}^N)} \leqslant\; C_{p'}\|\nabla\varphi\|_{L^{p'}(\mathbb{R}^N)}$$

which holds, when

$$\frac{N}{N-1} < p \leqslant \infty \Leftrightarrow 1 \leqslant\; p' < N.$$

This establishes the connection between the decomposition formulas on the Dirac basis and Hardy inegualities. Note however that, as we mentioned before, it is better to use the explicit formulas obtained above, without using Hardy inequalities and duality, to avoid the problems related with the extremal cases $p = 1, \infty$.

The problem of obtaining sharp properties of the function f guaranteeing that the infinite order expansion

$$f \;=\; \sum_{\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!}\int_{\mathbb{R}^N} f(x)x^\alpha\,dx\,D^\alpha\delta_0$$

holds true is open.

2.4 Scaling: A basic tool for computing asymptotics.

We have shown that, using multipliers and integration by parts, one can get sharp $L^p(\mathbb{R}^N)$-estimates and that this method applies as well for a number of nonlinear parabolic equations (see [[27]). However, obviously, in the nonlinear context, there is no explicit convolution formula for solutions allowing to obtain the asymptotic expansion. In this case scaling arguments are very useful although they give, in principle, only the first term of the expansion. Note also that, as pointed out in [28], the asymptotic behavior might be extremely complex.

Let us consider again the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \mathbb{R}^N \quad t > 0 \\ u(x, 0) = \varphi(x) & \mathbb{R}^N. \end{cases}$$

We know that

$$\varphi \in L^1(\mathbb{R}^N) \Rightarrow \|u(t)\|_p \leqslant C_p\, t^{-\frac{N}{2}(1-\frac{1}{p})} \|\varphi\|_{L^1(\mathbb{R}^N)}.$$

We now use the scaling argument. For, we introduce the rescaled family

$$u_\lambda(x, t) = \lambda^N u(\lambda x, \lambda^2 t), \quad \lambda > 0.$$

It is easy to see that, for each $\lambda > 0$, u_λ solves the heat equation as well:

$$\begin{cases} u_{\lambda,t} - \Delta u_\lambda = 0 & \mathbb{R}^N,\ 0 < t < 1 \\ u_\lambda(0) = \varphi_\lambda = \lambda^N \varphi(\lambda x). \end{cases}$$

At this point it is important to observe that the initial data of the rescaled family of solutions satisfy

$$\varphi_\lambda \longrightarrow \int_{\mathbb{R}^N} \varphi(x)\, dx\, \delta_0, \quad \lambda \to \infty.$$

On the other hand, the decay rate of the solution u in L^p yields uniform bounds on the rescaled family of solutions u_λ:

$$\|u_\lambda(t)\|_p \leqslant C_p\, t^{-\frac{N}{2}(1-\frac{1}{p})},\ 0 < t < 1.$$

This inequality holds uniformly on λ with a constant C_p independent of this parameter.

This estimate, together with the regularizing effect of the heat equation allows showing that, u_λ is relatively compact in $L^p(\mathbb{R}^N)$ for all $1 \leq p \leq \infty$, for each $t > 0$. In fact compactness holds in $C([\tau, 1]; L^p(\mathbb{R}^N))$, for all $0 < \tau < 1$. Let v be the limit of u_λ, after extraction of a suitable subsequence:

$$u_\lambda \to v, \quad \lambda \to \infty.$$

One can show that the limit v is the solution of

$$\begin{cases} v_t - \Delta v = 0 \quad \text{in}\, \mathbb{R}^N \times (0,1) \\ v(0) = \int_{\mathbb{R}^N} \varphi \, dx \, \delta_0. \end{cases}$$

By uniqueness of the fundamental solution $v = \int \varphi \, dx \, G$. Consequently the limit holds along the whole familiy $\lambda \to \infty$.

Then

$$u_\lambda\big|_{t=1} \overset{\lambda\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} \varphi dx \, G\big|_{t=1} \, in \quad L^p(\mathbb{R}^N),$$

for all $1 \le p \le \infty$. This turns to be equivalent to the fact that

$$||u(t)|| - \int_{\mathbb{R}^N} \varphi dx \, G||_{L^p(\mathbb{R}^N)} t^{\frac{N}{2}(1-\frac{1}{p})} \underset{t\to\infty}{\longrightarrow} 0 \, .$$

This is precisely the first term in the asymptotic expansion we got through the explicit representation formula of solutions.

Summarizing, we see that sharp L^p decay estimates, together with the regularizing effect of parabolic equations (to gain compactness), scaling arguments and the uniqueness of the limit characterized as the solution of the limit system, allow obtaining the first term of the asymptotic expansion of the solution as $t \to \infty$.

This method works for much more general problems (variable coefficients, nonlinearities, etc.) but fails to give a complete asymptotic expansion.

In fact, as we show by means of the explicit representation formula, when the initial datum φ satisfies the further property that $|x|\varphi(x) \in L^1(\mathbb{R}^N)$, then we gain an extra decay rate of the order of $t^{-1/2}$. The scaling argument above in itself does not yield this extra information.

We have also mentioned above that this method allows exhibiting, in some particular cases a much more complex asymtotic behavior of solutions. We shall briefly discuss this matter in the following subsection.

2.5 How Universal is the Law of Asymptotic Simplification ?

Let us consider now the Cauchy problem for a linear parabolic equation with variable coefficients:

$$\begin{cases} u_t \,-\, \text{div}(a(x)\nabla u) \,=\, 0 \quad \mathbb{R}^N, \, t>0 \\ u(0) \,=\, \varphi, \end{cases}$$

where

$$0 \,<\, \alpha \,\le\, a(x) \,\le\, \beta \,\le\, \infty \qquad \text{a.e.} \;\in \mathbb{R}^N.$$

For this equation, the argument above, based on mutiplying the equation by powers of the solution, and integration by parts, yields the same decay rate as for the constant coefficient heat equation. Namely, it follows that

$$||u(t)||_p \le C_p t^{-\frac{N}{2}(1-\frac{1}{p})} ||\varphi||_1.$$

Let us now perform the scaling argument. For, introduce the rescaled family of solutions:

$$u_\lambda = \lambda^N u(\lambda x, \lambda^2 t).$$

In this case they satisfy

$$\begin{cases} u_{\lambda,t} - \mathrm{div}(a(\lambda x)\nabla u_\lambda) = 0 \\ u_\lambda(0) = \varphi_\lambda = \lambda^N \varphi(\lambda x) \longrightarrow \int \varphi dx\, \delta_0. \end{cases}$$

Note that, in this case, due to the presence of the variable coefficient, the equation associated with u_λ depends on λ. In other words, the equation is not invariant under the scaling transformation. It is then important to analyze how does $a(\lambda x)$ behave as $\lambda \to \infty$. There are several cases to be distinguished:

- *Periodic case*: Assume that $a = a(x)$ is $[0,1]^N$-periodic i.e. it is periodic of period one in each of the N space variables. Then, $a_\lambda = a(\lambda x)$ is $\left[0, \frac{1}{\lambda}\right]^N$-periodic and

  $$a_\lambda \overset{\lambda\to\infty}{\longrightarrow} \bar{a} = \int_{[0,1]^N} a\, dy\,, \quad \lambda \to \infty.$$

  weakly-* in $L^\infty(\mathbb{R}^N)$.

  But, contradicting the first intuition, in this case the limit v of the rescaled solutions u_λ does not satisfy the equation

  $$v_t - \bar{a}\Delta v = 0.$$

  Indeed, the theory of Homogenization guarantees that the limit function solves the *homogenized* equation

  $$v_t - \mathrm{div}(A^* \nabla v) = 0,$$

  where A^* is a constant coefficient symmetric and elliptic matrix that does not coincide with $\bar{a}I$ except in the trivial case where the coefficient $a = a(x)$ is constant.

  For instance, in one space dimension ($N = 1$), the homogenized coefficient A^* is

  $$A^* = \left(\int_0^1 \frac{1}{a}\, dy\right)^{-1}$$

  which does indeed differ from the average $\bar{a}$ of a except when a is constant.

This can be easily seen in the context of the elliptic equation

$$-(\ a(\lambda x)u'_\lambda\ )' = f.$$

Indeed, explicit computations yield

$$a(\lambda x)u'_\lambda = -\int_{-\infty}^{x} f$$

and

$$u'_\lambda = -\frac{1}{a(\lambda x)}\int f \longrightarrow -\fint \frac{1}{a}\int_{-\infty}^{x} f.$$

Consequently,

$$u_\lambda \longrightarrow v\ \mathcal{D}'$$

where v solves

$$-a^* v'' = f$$

with

$$a^* = \left(\fint \frac{1}{a}\right)^{-1}.$$

But although the analysis of this case is not straightforward, the periodic coefficient case is an easy one. In fact, using Block waves one can get a complete asymptotic expansions of solutions as $t \to \infty$ (see [10] and [21]).

- *Asymptotically Constant Difussion:* Assume now that

$$a(x) = 1 + \epsilon(x), \quad \epsilon(x) \to 0,\ |x| \to \infty.$$

Then,

$$a(\lambda x) = 1 + \epsilon(\lambda x) \to 1, \quad \lambda \to \infty,\ \text{a.e. } x \in \mathbb{R}^N.$$

In this case the limit v of the rescaled solutions u_λ does indeed satisfy

$$\begin{cases} v_t - \Delta v = 0 \\ v(0) = \int_{\mathbb{R}^N} \varphi(x)\,dx\,\varphi_0, \end{cases}$$

and therefore

$$v = \int_{\mathbb{R}^N} \varphi(x)\,dx\,G\,.$$

We refer to [?] for a further analysis of this case.

So far, in all the examples, asymptotic simplification occurs and solutions behave as a gaussian process in a suitable homogeneous medium. But this is not true for all coefficients $0 < \alpha \leq a(x) \leq \beta < \infty$ since complexity arises as well. The following holds:
Lemma. ([28])

Given any sequence $\{f_j\}_{j\in\mathbb{N}}$*, bounded in* $L^\infty(\mathbb{R}^N)$*, we can construct a function* $g \in L^\infty(\mathbb{R}^N)$ *such that the weak-* accumulation points of the rescaled family*

$$g_\lambda(x) = g(\lambda x) \quad \text{as } \lambda \to \infty$$

contain the closure of the sequence $\{f_j\}_{j\in\mathbb{N}}$ *on the weak-* topology.*

In particular, for a suitable g*, theset of accumulation points of the family* $\{g_\lambda\}$ *may cover a ball in* $L^\infty(\mathbb{R}^N)$*.*

Proof. The Lemma can be easily proved by the "zooming method" that we describe briefly.

We first cut each function f_i in the annulus $2^{-j} < |x| < 2^j$ and then zoom it with a zooming factor $\lambda_{i\,j} > 0$. In this way we get the new function

$$f_{i\,j}(x) = f_i\left(\frac{x}{\lambda_{i\,j}}\right) \qquad 2^{-j}\lambda_{i\,j} < |x| < 2^j\lambda_{i\,j}.$$

We arrange all these annulae so that they become disjoint by choosing appropriate values of $\lambda_{i\,j}$ and then define the function g as being

$$g(x) = f_i\left(\frac{x}{\lambda_{i\,j}}\right) \quad \text{in each of the annulae.}$$

We then consider the rescaled family $g_\lambda = g(\lambda x)$. It is important to observe that, along the sequence $\lambda = \lambda_{i\,j}$, $g_\lambda \equiv f_i$. Therefore, the function f_i is included in the set of accumulation points of g_λ and this for all indexes i.

This concludes the proof of the Lemma.

∎

This has important consequences in the asymptotic behavior of solutions of parabolic equations with coefficients having the structure of the function g in the Lemma. Indeed, consider for instance the parabolic equation with variable density

$$\rho(x)\, u_t - \Delta u = 0.$$

The density ρ, according to the previous Lemma, can be chosen such that the set of accumulation points of the rescaled family ρ_λ contains any sequence $\{f_j\}$. The accumulation points of the rescaled family of solutions u_λ then solve the limiting equations

$$\begin{cases} f_j(x)\, v_t - \Delta v = 0 \\ v(0) = \int_{\mathbb{R}^N} \varphi\, \delta_0, \end{cases}$$

for $j \geq 1$.

Consequently, the set of accumulation points of the rescaled family u_λ contains the whole family of fundamental solutions G_j of these equations, i.e.

$$\lim_{\lambda\to\infty} u_\lambda \supset \left\{ \int \varphi\, dx\, G_1,\ \int \varphi\, dx\, G_2,\ \int \varphi\, dx\, G_3 \ \ldots \right\}.$$

This shows that, although for some important cases asymptotic simplication occurs, the large time asymptotic behavior may be quite complex in general. The problem of obtaining a complete asymptotic expansion to any order for general variable coefficient parabolic equations is still open. We refer to [28] for a deeper discussion of this issue.

3 The Wave Equation

3.1 The Dirichlet problem

Let us consider now the wave equation in a bounded domain Ω with Dirichlet boundary conditions:

$$\begin{cases} u_{tt} - \Delta u = 0 & \Omega \times (0,\infty) \\ u|_{\partial\Omega} = 0 & \\ u(0) = u_0,\ u_t(0) = u_1 & \text{in } \Omega. \end{cases}$$

It is a model for the vibrations of strings and membranes and also for the propagation of acoustic waves.

This system is well-posed in the energy space $H_0^1(\Omega) \times L^2(\Omega)$. Then, given $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ there exists a unique solution $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$. Moreover, the energy

$$E(t) = \frac{1}{2} \int_\Omega \left[|\nabla u(x,t)|^2 + u_t^2(x,t) \right] dx$$

is conserved in time, i.e.

$$E(t) = E(0),\ \forall t > 0\,.$$

This system exhibits the following features that are in opposition with those characterizing the heat equation:

- The system is purely conservative.
- The system lacks of smoothing properties.
- The system is time reversible.

The property of conservation of energy can be easily proved by the energy method. In this case, multiplying the equation by u_t and integrating by parts we get

$$\frac{dE}{dt} = \int_\Omega u_{tt}\, u_t + \nabla u \cdot \nabla u_t dx = \int_\Omega (u_{tt} - \Delta u)\, u_t\, dx = 0.$$

Solutions of the wave equation can be easily developed in Fourier series. Consider again the eigenvalue problem

$$\begin{cases} -\Delta\phi_j(x) = \lambda_i \phi_j(x) & \Omega \\ \phi_j\big|_{\partial\Omega} = 0 & j = 1, 2, \ldots \end{cases}$$

Then

$$u(x,t) = \sum_{j=1}^{\infty} \left(a_j \cos(\sqrt{\lambda_j}\, t) + \frac{bj}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}\, t) \right) \phi_j(x),$$

where $\{a_j\}$ and $\{b_j\}$ are the Fourier coefficients of the initial data

$$u_0(x) = \sum_{j=1}^{\infty} a_j \phi_j(x), \quad u_1(x) = \sum_{j=1}^{\infty} b_j \phi_j(x).$$

In other words:

$$u = \sum_{k=1}^{\infty} u_k(t)\, \phi_k(x)$$

where

$$\begin{cases} u_k'' + \lambda_k\, u_k = 0, & t > 0 \\ u_k(0) = a_k,\ u_k'(0) = b_k, & k = 1, \ldots, \infty \end{cases}$$

Consequently, the wave equation is equivalent to a system of infinitely many uncoupled harmonic oscillators.

3.2 The Damped Wave Equation

The system above is purely conservative. But in most physical systems friction is present. Friction or damping is also a very efficient way for stabilizing engineering systems. This was already observed by L. Maxwell in his work "Governors" on the dynamical properties of the steam-engine in the end of XIX-th century.

The damped wave equation reads as follows

$$\begin{cases} u_{tt} - \Delta u + d u_t = 0 & \Omega \times (0, \infty) \\ u = 0 & \partial\Omega \times (0, \infty) \\ u(0) = u_0,\ u_t(0) = u_1 & \Omega \end{cases}$$

where $d > 0$ is the damping coefficients.

The energy method yields the following *energy dissipation law*

$$\frac{dE}{dt} = -d \int_\Omega u_t^2 \leq 0 \ .$$

The following questions arise naturally: *Does* $E(t) \to 0, \, t \to \infty$ *?* and, if yes, *How Fast ?*

In this case the Fourier method yields the following representation formula

$$u = \sum_{k=1}^{\infty} u_k(t) \, \phi_k(x)$$

where the coefficients u_k obey the ODE

$$u_k'' + \lambda_k \, u_k + d u_k' = 0.$$

The two roots of the characteristic polynomial are

$$r_k = \frac{-d \pm \sqrt{d^2 - 4\lambda_k}}{2}.$$

Accordingly, two cases have to be distinguished:

- $d \leq 2\sqrt{\lambda_1} \ \Rightarrow \ \mathrm{Re}(r_k^{\pm}) = -\frac{d}{2}, \quad \forall k$
- $d > 2\sqrt{\lambda_1} \ \Rightarrow \ \exists$ finite number of real eigenvalues.

But, always, independently of the value of the damping constant $d > 0$ it follows that:

$$\max_k \mathrm{Re}(r_k) \geq \max_k \mathrm{Re}(r_k)\Big|_{d=2\sqrt{\lambda_1}} = -\sqrt{\lambda_1} \, .$$

This means that, among the class of constant dampers, the one that produces the best decay rate for the energy is $d = 2\sqrt{\lambda_1}$ in which case the energy decays exponentially with a rate $-2\sqrt{\lambda_1}$.

In particular it is not true, as one could expect in a first approach, that increasing the damping constant necessarily yields a better decay rate. This occurs for $0 \leq d \leq d = 2\sqrt{\lambda_1}$ but it is not longer true for $d > 2\sqrt{\lambda_1}$. This phenomenon is refered to as "overdamping" in the engineering literature.

One can do much better by means of non-constant damping coefficients. But the corresponding spectral problem is not easy to deal with, since it is not self-adjoint.

3.3 Boundary damping

An interesting example is the one related with the wave equation with boundary damping which leads to a new phenomenon: the *vanishing waves.*

Consider the $1-d$ wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1,\ t > 0 \\ u(0,t) = 0, & t > 0 \\ u_x(1,t) + u_t(1,t) = 0, & t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & 0 < x < 1. \end{cases}$$

The energy dissipation law now reads as follows:

$$\frac{dE}{dt}(t) = -u_t^2(1,t)$$

In this particular case not only solutions decay exponentially as $t \to \infty$. But, in fact, solutions vanish in finite time ($T = 2$). This is closely related with the fact that the spectrum of the system is empty and indicates that the evolution process under consideration is highly irreversible.

To prove this property of vanishing of waves it is convenient to decompose the d'Alembert operator in two first order transport equations:

$$u_{tt} - u_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x)u = 0$$

A simple analysis[‡] of the behavior of solutions along characteristic lines allows showing that the vanishing of waves property is indeed true in time $T = 2$. Of course, at this level, the fact that in the dissipative boundary condition $u_x(1,t) + u_t(1,t) = 0$ one of these transport operators appears plays a key role.

But this example is impossible to reproduce for general equations or higher dimensional situations and it is very specific to the $1-d$ wave equation with constant coefficients. In fact it does not even hold for the $1-d$ wave equation with variable coefficients (see [7]).

The boundary conditions for which solutions vanish in final time are often refered to as transparent boundary conditions since, as the analysis of this $1-d$ models shows, waves, when reaching the dissipative boundary condition are not reflected at all. Transparent boundary conditions are very useful in numerics since they allow reproducing in a bounded domain the properties of the Cauchy problem in the whole space. The fact of being able to reduce the problem to solving a system in a bounded domain, of course, reduces significantly the computational cost of the numerical method.

[‡]This is an interesting exercise. Note that in particular this leads to a strongly irreversible semigroup. Note also that imposing similar boundary conditions in both extremes of the space interval one can achieve the vanishing of waves in time $t = 1$.

3.4 Internal damping

There is an intermediate damping mechanism: the one in which the damping term is localized in a subdomain ω of the domain Ω where the equation holds:

$$\begin{cases} u_t - \Delta u + 1_\omega u_t = 0 \\ u\big|_{\partial\Omega} = 0 \\ u(0) = u_0, \qquad u_t(0) = u_1. \end{cases}$$

Here and in the sequel 1_ω denotes the characteristic function of the set ω.

In this case the energy dissipation law reads

$$\frac{dE}{dt} = -\int_\omega u_t^2\, dx.$$

Theorem. *The energy decays exponentially provided ω is a neighborhood of a subset of the boundary of the form*

$$\Gamma(x_0) = \{x \in \partial\Omega : (x - x_0) \cdot n(x) > 0\},$$

for some $x_0 \in \mathbb{R}^N$.

More precisely, there exist positive constants $C_{\Omega,\omega}$ and $\alpha_{\Omega,\omega}$, depending on Ω and ω such that

$$E(t) \leqslant C_{\Omega,\omega} e^{-\alpha_{\Omega,\omega} t} E(0),$$

for every solution of the system.

Remark.

- Here $n(x)$ denotes the unit normal vector to the domain Ω at the boundary point $x \in \partial\Omega$.
- The subset $\Gamma(x_0)$ of the boundary is constituted by the points in which the ray going from the reference point x_0 to x exits the domain Ω.
- In the context of the linear wave equation under consideration a sharp necessary condition for the exponential decay was given in [1] in Geometric Optics terms. This is the so called Geometric Control Condition and reads as follows: The exponential stability property holds if all rays of Geometric Optics, after possibly bouncing on the boundary of the domain Ω, reach the subdomain ω where the damping term is effective.
- Similar results hold for semilinear wave equations as well, see [8] and [33].

Sketch of the proof:

- Step 1: It suffices to check that there exist $C, T > 0$:

$$E(0) \leqslant C \int_0^T \int_\omega u_t^2 dxdt,$$

for every solution.

This inequality, refered often as *observability inequality*, indicates that the dissipated energy is proportional to the energy within the system.

This observability inequality, combined with the energy dissipation law yields

$$E(T) - E(0) = -\int_0^T \int_\omega u_t^2 dxdy \leqslant -\frac{1}{C}E(0),$$

and consequently,

$$E(T) \leqslant (\frac{C-1}{C})E(0).$$

Accordingly, the semigroup at time $t = T$, $S(T)$, is a contraction.

The semigroup property then yields the exponential decay. Indeed, when $t = kT$ it follows that

$$\|S(t)\| = \|S(kT)\| \leqslant \gamma^k = e^{k\log\gamma} = e^{-|\log\gamma|t/T}.$$

- Step 2: We now proceed to the prove the observability inequality.

The difference between the damped and the conservative wave equation being the damping term itself, it suffices to prove the inequality for the conservative wave equation.

$$\begin{cases} u_{tt} - \Delta u_\lambda = 0 & \Omega \times (0,T) \\ u\big|_{\partial\Omega} = 0 & \\ u(0) = u_0, u_t(0) = u_1 & \Omega. \end{cases}$$

The question is now whether

$$E(0) \leqslant C \int_0^T \int_\omega u_t^2 dxdt.$$

This inequality does indeed hold and can be proved using the so called multiplier method. It consists in multiplying the equation by u, u_t, $(x - x_0)\cdot\nabla u$,... and integrating by parts to later combine the identities one obtains (see [17] and [19] for a systematic description of this method).

This method is very much the same as that developed by Pokhozhayev in the context of elliptic equations [§] and first introduced by F. Rellich [24]. Pokhozhayevs identity

[§]For further information on Pokhozhayev's identity see the web page `http://equinox.unr.edu/homepage/jm/utah03/`

allows getting connections between the total energy of solutions of elliptic problems and its energy concentrated on the boundary.

Let us recall how this identity may be established for the eigenvalue problem

$$-\Delta u = \lambda u \quad \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Multiplying the equation by $x \cdot \nabla u$ (we assume without loss of generality that $x_0 = 0$) we get

$$\lambda \int_\Omega u x \cdot \nabla u dx = -\frac{N\lambda}{2} \int_\Omega u^2 dx$$

and

$$\begin{aligned} -\int_\Omega \Delta u x \cdot \nabla u dx &= \int_\Omega \nabla u \cdot \nabla(x \cdot \nabla u) dx - \int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 (x \cdot n) d\sigma \\ &= \int_\Omega \left[|\nabla u|^2 + x \cdot \nabla \frac{|\nabla u|^2}{2} \right] - \int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 (x \cdot n) d\sigma \\ &= \left(1 - \frac{N}{2}\right) \int_\Omega |\nabla u|^2 - \frac{1}{2} \int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 (x \cdot n) d\sigma. \end{aligned}$$

Moreover, multiplying the equation by u it follows that

$$\int_\Omega |\nabla u|^2 = \lambda \int_\Omega u^2 = \lambda$$

provided we normalized the eigenfunctions by the condition $\left(\int_\Omega u^2 = 1\right)$. Combining these identities we get

$$\left(1 - \frac{N}{2}\right) \lambda - \frac{1}{2} \int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 (x \cdot n) d\sigma = -\frac{N}{2}\lambda.$$

Thus

$$\lambda = \frac{1}{2} \int_{\partial\Omega} (x \cdot n) \left|\frac{\partial u}{\partial n}\right|^2 d\sigma \leqslant \frac{1}{2} \int_{\Gamma(x_0)} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma.$$

Consequently, it follows that

$$\boxed{\int_\Omega |\nabla u|^2 dx \leqslant \frac{1}{2} \int_{\Gamma(x_0)} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma.}$$

It is important to note that this inequality holds independently of the frequency λ, and it does indeed guarantee that the energy of the eigenfunctions is uniformly bounded by the energy concentrated on the subset of the boundary $\Gamma(x_0)$.

As a consequence of this, with some extra work, taking into account that the subdomain ω is a neighborhood of Γ_{x_0}, it can also be proved that

$$\boxed{\int_\Omega |\nabla u|^2 dx \leqslant C \int_\omega |\nabla u|^2 \, dx,}$$

with a constant C which is independent of the eigenfunction.

The same occurs for the wave equation when T is large enough, namely when $T > \mathrm{diam}(\Omega \setminus \omega)$.

■

This Theorem shows that, when ω is a neighborhood of a large enough subset of the boundary, the exponential decay property holds. As a complement to this result it is important to observe that, in general, for any non-empty open subset ω of Ω the property of the exponential decay of the energy is no longer true but still the energy of every solution tends to zero without uniform decay rate. The fact that the energy of each solutions tends to zero can be proved as an application of La Salle's invariance principle.

3.5 Damping localized on narrow sets

We have discussed the case where the damping is localized in an open non-empty subset of the domain Ω. But the question makes also sense whatever ω is. It is for instance natural to analyze the case where ω is a measurable set of positive measure. In this case, a Lyapunov type argument allows showing that the energy of each individual trajectory tends to zero. The question of whether exponential decay holds under suitable additional assumptions is open in the context of damping sets of positive measure.

Applying La Salle's invariance argument and using the energy E of the system as Lyapunov function one can show that every solution tends to zero if and only the following uniqueness or unique continuation property holds: *The only solution of the damped equation*

$$\begin{cases} u_{tt} - \Delta u + 1_w u_t = 0 \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

such that $u_t = 0$ *in the damping region* $\omega \times (0, \infty)$ *is the trivial one* $u \equiv 0$.

However, taking into account that the damping term vanishes under the condition that $u_t = 0$ in the damping region $\omega \times (0, \infty)$, it is in fact equivalent to proving the same unique continuation property for the undamped equation

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u\big|_{\partial\Omega} = 0. \end{cases}$$

The question is then whether $u_t = 0$ in $\omega \times (0, \infty)$ for the solutions of the conservative wave equation implies that $u \equiv 0$.

To analyze this question it is convenient to develop solutions in Fourier series. In fact we set $v = u_t$. Obviously, v is also a solution of the Dirichlet problem for the wave equation and $v = 0$ in $\omega \times (0, \infty)$. The Fourier representation formula yields

$$v(x,t) = \sum a_k^{\pm} e^{\pm i \sqrt{\lambda_k} t} \phi_k(x),$$

where the sum runs along the whole sequence of eigenpairs of the Laplace operator.

We now use the orthogonality propertiy of complex exponentials in infinite time

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\lambda t}e^{-i\mu t}\,dt = \begin{cases}1 & \text{if } \lambda=\mu\\ 0 & \text{if } \lambda\neq\mu.\end{cases}$$

Then, the fact that $v=0$ in $\omega\times(0,\infty)$ implies

$$\int_0^\infty\int_\omega v^2(x,t)\,dx\,dt\equiv 0$$

and the following alternative holds: Either $v\equiv 0$ or $\phi_k^2=0$ in ω for some k.

However, the second possibility can inmediately be excluded since the eigenfunctions of the Laplacean are analytic and therefore they can not vanish in a set of positive measure without being trivially zero everywhere. Thus, we conclude that $u_t\equiv v\equiv 0$. Therefore, $u=u(x)$ and it solves the elliptic problem

$$\begin{cases}-\Delta u=0\\ u\big|_{\partial\Omega}=0.\end{cases}$$

The only solution of this elliptic equation being $u\equiv 0$, we deduce that the desired unique continuation property holds.

As a consequence of this analysis it follows that for damping sets ω of positive measure one can guarantee that all solutions tend to zero. The problem of getting conditions under which the decay rate is uniform and exponential is open, as we mentioned before.

Similar problems arise when the damper is localized in manifolds of lower dimension (see for instance [15] and [20]). One of the most classical examples is the case where the damper is localized in one single point x_0 of the domain.

Let us consider this problem in $1-d$ the domain Ω under consideration being $\Omega=(0,\pi)$:

$$\begin{cases}u_{tt}-u_{xx}+u_t(x_0,t)\delta_{x_0}=0\\ u(0,t)=u(\pi,t)=0.\end{cases}$$

In this case, the Fourier series development of solution reads

$$u=\sum a_k^{\pm}e^{\pm ikt}\sin(kx).$$

The damping being localized at the point $x_0\in\Omega$ the question we analyze is closely related to how much of the energy of vibrations do we estimate through the observation of $\displaystyle\int_0^T u_t^2(x_0,t)\,dt$?

The answer is easy in this case. Indeed, by taking $T = 2\pi$ and using the orthogonality properties of trigonometric polynomials it follows that

$$\int_0^{2\pi} u_t^2(x_0,t)\,dt = \int_0^{2\pi} \left|\sum \pm a_k^{\pm} ike^{\pm ikt}\sin(kx_0)\right|^2 dt = 2\pi \sum_k \left|a_k^{\pm}\right|^2 k^2 \sin^2(kx_0).$$

According to this we have to exclude the cases where x_0/π is rational or not. Indeed,

• $x_0/\pi \in \mathbb{Q} \Rightarrow \exists\, k\colon \sin(kx_0) = 0$.

In this case there are Fourier components that we do not see at all! Consequently, the observed quantity does not constitute a norm in the space of solutions. In what concerns the damped equation, this means that there are solutions that are not damped at all, whose energy remains constant in time.

• $x_0/\pi \in \mathbb{Q}^c \Rightarrow \gamma_k = \sin(kx_0) \neq 0$ for all k. In this case, for all $T \geq 2\pi$ we obtain the inequality

$$\sum_k \gamma_k^2 |a_k^{\pm}|^2 k^2 \leq C \int_0^T |u_t(x_0,t)|^2\,dt.$$

However, the asymptotic behavior of the weights γ_k depends on the class of irrationality in which $\frac{x_0}{\pi}$ lies. In particular,

• When $\frac{x_0}{\pi}$ is an algebraic number of degree 2,

$$|\gamma_k| \sim c/|k|$$

• By the contrary, if $\frac{x_0}{\pi}$ is a Liouville number

$$|\gamma_k| \to 0\,, \qquad \text{exponentially as } k \to 0\,.$$

According to this, in the best case, for an optimal choice of x_0, i.e. when x_0/π is an algebraic number of degree 2, we loose one derivative on the observability inquality in the sense that we get

$$\|(u_0,u_1)\|^2_{L^2(0,\pi)\times H^{-1}(0,\pi)} \leq C \int_0^{2\pi} u_t^2(x_0,t)\,dt,$$

instead of the sharp inequality

$$E(0) \leq C \int_0^{2\pi} u_t^2(x_0,t)\,dt,$$

one could expect by comparison with the case where the damping is effective in an open subset of the domain. The latter is never true, whatever the choice of x_0 is.

This weakened observability estimates lead to polynomial decay rates for smooth solutions of the damped wave equation (see [15]). In $1-d$ these results were explained in terms of ray properties in [20]. There it was shown that one could concentrate solutions of the wave equation along rays, cancelling each other, so that the observability inequelity above with a defect of one derivative becomens sharp.

In $1-d$ wave equations with variable coefficients behave very much the same as the constant coefficient one discussed above. However, in that case, one can not rely on the property of orthogonality of trigonometric polynomials. The classical *Ingham inequality* plays the same role atthis level (see [32]).

The problem of combining Geometric Optics tools with others like diophantine approximations or ergodicity theory to obtain polynomial decay rates when the damping acts on narrow sets in several space dimensions is completely open.

The same can be said about the case where the damper is located in moving points. We refer to [3] for the analysis of the approximate control property for the heat equation. But nothing is known for in the context of stabilization of the wave equation.

4 Convergence to equilibrium

4.1 Introduction

We have seen that, when discretizing an evolution equation of Navier-Stokes or Burgers type in time, in each step of the time iteration, we have to solve a nonlinear elliptic equation. The same happens when we address the solution of evolution equations via nonlinear semigroup theory, as we will see below.

However, the underlying elliptic problems have an importance that go beyond these modeling considerations. Indeed, there are many cases in which, in order to simplify the model under consideration, the evolution equation is substituted by a stationary equation. On the scope of the equations analyzed in this course, this implies going from a parabolic equation to an elliptic equation.

This entails a very important simplification also from the point of view of numerics, since the time dependency disappears.

There is a reason behind this simplification that is valid for several equations: as $t \to \infty$, the solutions undergo an asymptotic simplification that makes them similar to a solution of the corresponding stationary equation. In this section, we will analyze some easy cases in which this fact can be rigorously proven. Obviously, in practice, this possibility of simplifying the model to make it stationary is used even in cases in which this is not rigorously justified. Of course, this can be a reason to nullify the obtained results. It's precisely for this reason that it is important to have a knowledge of the basic techniques that allow to justify this simplification at the level of modeling.

4.2 Finite dimensional gradient systems

We will first consider a finite dimensional gradient system:

$$\begin{cases} x'(t) + \nabla H(x(t)) = 0, & t > 0 \\ x(0) = 0, \end{cases} \tag{1}$$

where

$$H : \mathbb{R}^N \to \mathbb{R} \tag{2}$$

is a C^2 convex function, attaining its minimum in a unique point $x^* \in \mathbb{R}^N$:

$$H(x^*) = \min_{x \in \mathbb{R}^N} H(x). \tag{3}$$

Obviously, we have that:

$$\nabla H(x^*) = 0, \tag{4}$$

and thus x^* is a stationary solution of (1). Actually, x^* is the unique stationary solution of (1), since the necessary and sufficient condition for being so is being a critical point of H, and by the convexity of H there is only one such point: x^*.

Multiplying in (1) by $x'(t)$ we have:

$$\mid x'(t) \mid^2 + \langle \nabla H(x(t)), x'(t) \rangle = 0. \tag{5}$$

Where $| \cdot |$ denotes the euclidean norm in $\mathbb{R}^N$ and $\langle \cdot, \cdot \rangle$ the associated scalar product. On the other hand:

$$\langle \nabla H(x(t)), x'(t) \rangle = \frac{d}{dt} H(x(t)).$$

And thus we deduce the identity:

$$\frac{d}{dt} H(x(t)) = - \mid x'(t) \mid^2, \tag{6}$$

from which we obtain that:

$$H(x(t)) + \int_0^t \mid x'(s) \mid^2 ds = H(x_0). \tag{7}$$

In particular:

$$H(x(t)) \leq H(x_0). \tag{8}$$

Now, assuming that H is coercive, i.e.:

$$\lim_{|x| \to \infty} H(x) = \infty, \tag{9}$$

which is a natural hypothesis when minimizing the functional H, we deduce that the trajectory $t \to x(t)$ is bounded.

We define now the ω-limit set:

$$\omega(x_0) = \{ y_0 \in \mathbb{R}^N : \exists t_j \to \infty,\, x(t_j) \to y_0 \} \tag{10}$$

which is non-empty.

From La Salle's invariance principle¶, from the dissipation lay (6), it's easy to check that the solution $y(t)$ of (1) with initial data y_0 is such that $\nabla H(y_0) = 0$. From the uniqueness of x^* as the critical point of H, we deduce that $y_0 = x^*$. This proves that:

$$\omega(x_0) = \{x^*\} \tag{11}$$

and thus:

$$x(t) \to x^*,\, t \to \infty. \tag{12}$$

The result that we have just proven shows that, under very general conditions on the potential H, every solution of (1) converge to the unique equilibrium solution x^*. This allows to justify the evolution model (1) by the stationary model (4). However, this must be carried out carefully, since the proof of convergence that we have given does not give any estimate of the speed of convergence.

Under additional hypotheses on the potential H, we can also estimate the speed of convergence. Given the solution $x(t)$ of (1) and the stationary solution x^* of (4), we consider the difference:

$$y(t) = x(t) - x^*.$$

We thus have:

$$\begin{aligned} y'(t) &= -\Big[\nabla H(x(t)) - \nabla H(x^*)\Big] \\ &= -\Big[\nabla H(y(t)) + x^*) - \nabla H(x^*)\Big]. \end{aligned}$$

Taking the scalar product with $y(t)$ in this equation, we deduce that:

$$\frac{1}{2}\frac{d}{dt} \mid y(t) \mid^2 = -\Big[\langle \nabla H(y(t) + x^*) - \nabla H(x^*),\, y(t)\rangle\Big].$$

If H is of class C^2, using the Taylor expansion we deduce that:

$$\langle \nabla H(y(t) + x^*) - \nabla H(x^*),\, y(t)\rangle = \langle D^2 H(\xi(t)) y(t),\, y(t)\rangle,$$

where $D^2 H$ denotes the Hessian matrix of H.

Assuming furthermore that H is strictly uniformly convex, we deduce the existence of a constant $\alpha > 0$ such that:

$$D^2 H(\xi) \geq \alpha I,\, \forall \xi \in \mathbb{R}^N, \tag{13}$$

¶ $H(x(t))$ is bounded below and non-decreasing. It thus has a limit $\lim_{t\to\infty} H(x(t)) = L$. On the other hand, since $x(t_j) \to y_0$, using the semigroup property, $x(t_j + t) \to y(t)$. Since $H(x(t_j + t)) \to L\, j \to \infty$ for every $t > 0$, we deduce that $H(y(t)) = L$ for every $t > 0$. Applying the energy identity (6) we deduce that $y'(t) = 0$, which is equivalent to $y(t) \equiv y_0$. Since the unique stationary solution is x^*, we deduce that $y_0 = x^*$

and thus:

$$\langle \nabla H(y(t)+x^*) - \nabla H(x^*),\, y(t)\rangle \geq \alpha \mid y(t) \mid^2,$$

in other words:

$$\frac{1}{2}\frac{d}{dt} \mid y(t) \mid^2 \leq -\alpha \mid y(t) \mid^2, \tag{14}$$

from which we obtain the exponential convergence:

$$\mid y(t) \mid \leq e^{-\alpha t} \mid y_0 \mid = e^{-\alpha t} \mid x_0 - x^* \mid . \tag{15}$$

4.3 The linear heat equation

We now consider the linear heat equation:

$$\begin{cases} u_t - \Delta u = f(x) & \text{in} \quad \Omega,\, t > 0 \\ u = 0 & \text{in} \quad \partial\Omega,\, t > 0 \\ u(x,\, 0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{16}$$

in a bounded and regular domain Ω of $\mathbb{R}^N$, where $f = f(x)$ is an external source independent of t.

By the usual methods (Galerkin, semigroup theory,...) it's easy to shows that if $u_0 \in L^2(\Omega)$ and $f \in H^{-1}(\Omega)$, then there is a unique solution to (16) in the class $u \in C\big([0,\, \infty);\, L^2(\Omega)\big) \cap L^2_{loc}\big(0,\, \infty;\, H^1_0(\Omega)\big)$.

On the other hand, the energy identity yields:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega \mid \nabla u \mid^2 dx = \int_\Omega f u dx \tag{17}$$

from which we deduce that:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega \mid \nabla u \mid^2 dx \leq \frac{\varepsilon}{2}\int_\Omega \mid \nabla u \mid^2 dx + \frac{1}{2\varepsilon} \parallel f \parallel^2_{H^{-1}(\Omega)},$$

for every $\varepsilon > 0$. Taking, for example, $\varepsilon = 1$, we deduce that:

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega \mid \nabla u \mid^2 \leq \parallel f \parallel^2_{H^{-1}(\Omega)} . \tag{18}$$

Applying the Poincaré inequality, that ensures the existence of a constant $c(\Omega) > 0$ such that:

$$\int_\Omega \mid \nabla \varphi \mid^2 dx \geq c(\Omega)\int_\Omega \varphi^2 dx,\ \forall \varphi \in H^1_0(\Omega), \tag{19}$$

we obtain the energy inequality:

$$\frac{d}{dt}\int_\Omega u^2 dx + c(\Omega)\int_\Omega u^2 dx \leq \parallel f \parallel^2_{H^{-1}(\Omega)}, \tag{20}$$

from which we deduce the bound:

$$\int_\Omega u^2(x,t)dx \le e^{-c(\Omega)t}\int_\Omega u_0^2(x)dx + \frac{(1-e^{-c(\Omega)t})}{c(\Omega)} \parallel f \parallel^2_{H^{-1}(\Omega)} . \tag{21}$$

We have thus proven that:

$$u \in L^\infty(0,\infty; L^2(\Omega)), \tag{22}$$

in other words, we have proven that the trajectory $\{u(t)\}_{t\ge 0}$ is bounded in $L^2(\Omega)$.

Once we know that the trajectory is bounded, we can consider its asymptotic behavior as $t \to \infty$. If the limit of $u(t)$ as $t \to \infty$ exists, it's expected to be a stationary solution, independent of t, of the equation. It will thus be a solution of the elliptic equation:

$$\begin{cases} -\Delta u = f & \text{en} \quad \Omega \\ u\Big|_{\partial\Omega} = 0. \end{cases} \tag{23}$$

Classical variational theory ensures the existence and uniqueness of a solution $u^* \in H_0^1(\Omega)$ to (23).

Let's now prove that:

$$u(t) \longrightarrow u^*, \text{ exponentially in } L^2(\Omega),\ t \longrightarrow \infty. \tag{24}$$

Indeed, we define:

$$v(t) = u(t) - u^* \tag{25}$$

which is the solution to the homogeneous heat equation:

$$\begin{cases} v_t - \Delta v = 0 & \text{en} \quad \Omega,\ t>0 \\ v = 0 & \text{en} \quad \partial\Omega,\ t>0 \\ v(x,0) = v_0(x) = u_0(x) - u^*(x) & \text{en} \quad \Omega. \end{cases} \tag{26}$$

In this case, the energy identity ensures that:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 dx + \int_\Omega |\nabla v|^2\, dx = 0. \tag{27}$$

Applying the Poincaré inequality, as in the previous arguement, we deduce that:

$$\parallel v(t) \parallel_{L^2(\Omega)} \le e^{-c(\Omega)t} \parallel u_0 - u^* \parallel_{L^2(\Omega)}, \tag{28}$$

which gives the exponential rate of convergence that we previously claimed.

In fact, (28) gives an explicit exponential decaying rate, of the order of the Poincaré constant $c(\Omega)$.

The energy method that we have just presented allows to obtain convergence to equilibrium rates for a wide class of linear and nonlinear parabolic equations, and even for dissipative wave equations. In this way, we can for example also address the case of the parabolic p-laplacian equation:

$$u_t - \left(\mid \nabla u \mid^{p-2} \nabla u\right) = 0. \tag{29}$$

Note, however, that we don't always obtain an exponential decaying rate, as this rate can possibly be polynomial.

For example, in the case of equation (29) with Dirichlet boundary conditions, the energy method yields the identity:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mid u \mid^2 dx + \int_\Omega \mid \nabla u \mid^p dx = 0.$$

In this case, the Poincaré inequality ensures the existence of a constant $c_p(\Omega) > 0$ such that:

$$\int \mid \nabla u \mid^p dx \geq c_p(\Omega) \int_\Omega \mid u \mid^p dx,$$

from which we obtain the estimate:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + c_p(\Omega)\int_\Omega \mid u \mid^p dx \leq 0.$$

Assuming that $p > 2$ and applying Hölder's inequality, we deduce that:

$$\int_\Omega u^2 dx \leq \left(\int_\Omega \mid u \mid^p dx\right)^{2/p} \mid \Omega \mid^{(p-2)/p}.$$

From these two previous inequalities we deduce that:

$$\frac{d}{dt}\int_\Omega u^2 dx + \frac{2c_p(\Omega)}{\mid \Omega \mid^{(p-2)/2}}\left(\int_\Omega \mid u \mid^2 dx\right)^{p/2} \leq 0.$$

And thus:

$$\int_\Omega u^2 dx \leq \left[\frac{(p-2)}{2}\left(\frac{2c_p(\Omega)}{\mid \Omega \mid^{(p-2)/2}} t + \frac{2\int_\Omega u_0^2 dx}{(p-2)}\right)^{(2-p)/2}\right]^{2/(p-2)} \leq c_p(\Omega) t^{-2/(p-2)},$$

that is, we obtain a polynomial decay rate with exponent $2/(p-2)$. As $p \nearrow \infty$, this convergence rate weakens, which reflects the increasing weakness of the dissipative effect of the p-laplacian. However, as $p \searrow 2$, the polynomial decaying rate $2/(p-2)$ goes to infinity, which reflects the fact that for the linear equation case $p = 2$ the decaying rate is exponential.

Let's go back to the linear equation (16). In (28) we have obtained an exponential decaying rate with constant $c(\Omega)$, the Poincaré constant. In order to analyze more carefully the exponential behavior of the solutions, it's convenient to use Fourier series expansions. We will need an orthogonal basis of $L^2(\Omega)$, $\{\varphi_j\}_{j\geq 1}$, given by eigenfunctions of the laplacian:

$$\begin{cases} -\Delta\varphi_j = \lambda_j\varphi_j & \text{in} \quad \Omega \\ \varphi_j = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

and associated to the corresponding sequence of eigenvalues:

$$0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N \leq \ldots \longrightarrow \infty.$$

We will give the Fourier expansion of the datum u_0 and f of (16).

We obtain:

$$f = \sum_{j\geq 1} f_j\varphi_j;\ u_0 = \sum_{j\geq 1} u_{0,\,j}\varphi_j, \tag{30}$$

the solution will have the form:

$$u(x,\,t) = \sum_{j\geq 1} u_j(t)\varphi_j. \tag{31}$$

The search for the solution is reduced to the computation of its Fourier coefficients. These are completely determined by the equations:

$$\begin{cases} u_j' + \lambda_j u_j = f_j, & t > 0 \\ u_j(0) = u_{0,\,j}, & j \geq 1 \end{cases} \tag{32}$$

which can be solved explicitly

$$u_j(t) = u_{0,\,j}e^{-\lambda_j t} + \frac{(1 - e^{-\lambda_j t})}{\lambda_j} f_j,\ j \geq 1. \tag{33}$$

From this expression, we deduce immediately that:

$$u_j(t) \longrightarrow \frac{f_j}{\lambda_j},\ t \longrightarrow \infty,\ \forall j \geq 1. \tag{34}$$

A more careful analysis shows that:

$$u(\cdot,\,t) \to_{t\to\infty} \sum_{j\geq 1} \frac{f_j}{\lambda_j}\varphi_j(x), \ \text{ en } L^2(\Omega). \tag{35}$$

This coincides with the result that is obtained using the energy method, since:

$$\sum_{j\geq 1} \frac{f_j}{\lambda_j}\varphi_j(x), \tag{36}$$

is precisely the Fourier series expansion of the stationary solution u^* of (23).

In fact, this analysis gives that:

$$\| u(t) - u^* \|_{L^2(\Omega)} \leq e^{-\lambda_1 t} \| u_0 - u^* \|_{L^2(\Omega)}, \tag{37}$$

being $\lambda_1 > 0$ the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$.

Actually, this estimate concides with the one obtained in (28), since the Poincaré constant $c(\Omega)$ coincides with λ_1, i.e.:

$$c(\Omega) = \lambda_1.$$

This is so because, by the mini-max principle, the first eigenvalue λ_1 is characterized as the minimum of the Rayleigh quotient:

$$\lambda_1 = \min_{\psi \in H_0^1(\Omega)} \frac{\int_\Omega | \nabla\psi |^2 \, dx}{\int_\Omega \psi^2 dx}.$$

And thus λ_1, the first eigenvalue, is the biggest constant satisfying the inequality:

$$\lambda_1 \int_\Omega \psi^2 dx \leq \int_\Omega | \nabla\psi |^2 \, dx, \; \forall \psi \in H_0^1(\Omega).$$

In this section we have proven that the solutions to the heat equation converge to a stationary solution, when the external source f is independent of time. The same type of arguments yield, for example, that if $f = f(x,t)$ depends periodically (and with period τ) of time t, then there is a unique τ-periodic solution $u^* = u^*(x,t)$ of:

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \Omega, \, 0 < t < \tau \\ u = 0 & \text{in} \quad \partial\Omega, 0 < t < \tau \\ u(0) = u(\tau) \end{cases}$$

so that, for any solution of the initial value problem:

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \Omega, \quad t > 0 \\ u = 0 & \text{in} \quad \partial\Omega, \quad t > 0 \\ u(x,\, 0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

we have that:

$$\| u(t) - u^*(t) \|_{L^2(\Omega)} \to 0, \, t \to \infty$$

exponentially. The details of this proof are left to the interested reader.

In this section and the last one we have shown some examples in which the solutions of the evolution equation, as $t \to \infty$, are asymptotically simplified when converging to a stationary state. In some cases, this allows to substitute the evolution model by a stationary one, but this entails some risk unless an a priori quantitative study of the convergence rate as $t \to \infty$ is carried out.

In practice, there are many cases in which a stationary model is adopted without a rigorous proof of the asymptotic convergence of the solutions of the evolution equation as $t \to \infty$. This is a constant in the field of numerical analysis in which, frequently, we are forced to use methods without a rigorous proof of convergence, except for a few simplified cases.

4.4 Gradient systems and descent methods

In section (4.2) we have proven that, for gradient systems of the form:

$$\begin{cases} x' + \nabla H(x) = 0, & t > 0 \\ x(0) = x_0, \end{cases} \tag{38}$$

and under adequate conditions of convexity and coercivity of the potential H, the solutions of (38) converge, as $t \to \infty$, to the stationary solution x^*:

$$\nabla H(x^*) = 0. \tag{39}$$

We can interpret this result, as we have been doing until now, as the asymptotic simplification that allows us to pass from (38) to (39).

However, this can also be interpreted in a different way. Indeed, under the convexity and coervicity hypotheses, the functional H has a unique critical point x^*, which is also the global minimum, solution of (39). The fact that the solutions of (38) converge, as $t \to \infty$, to this critical point, allows us to interpret the evolution equation (38) as a way of approximating the minimum of H. The equation (38) can thus be understood as a continuous in t algorithm for the approximation of the minimum of H.

In fact, it is a descent algorithm because, as it was proven in (6), the energy identity is satisfied:

$$\frac{d}{dt} H(x(t)) = - \mid x'(t) \mid^2, \tag{40}$$

which proves that $H(x(t))$ decreases as t increases. The differential equation (38) is thus a continuous mechanism for the minimization of the functional H that drives the trajectory from the initial point x_0 of energy $H(x_0)$ to x^*, the point of minimum energy $H(x^*)$, in a monotonic manner.

The same type of algorithm can be reproduced discretely in time. In order to do so, we introduce a time discretization of the equation (38) with step Δt:

$$\frac{x^{k+1} - x^k}{\Delta t} = -\nabla H(x^{k+1}) \tag{41}$$

which can also be written as:

$$x^{k+1} + \Delta t \nabla H(x^{k+1}) = x^k, \tag{42}$$

and also as:

$$x^{k+1} = (I + \Delta t \nabla H)^{-1}(x^k). \tag{43}$$

The mapping $h = (I + \Delta t \nabla H)^{-1}$ is well defined,. Indeed, in order to solve:

$$h(x) = y \Leftrightarrow x + \Delta t \nabla H(x) = y \tag{44}$$

it's enough to minimize the functional:

$$J(x) = \frac{1}{2} \mid x \mid^2 + \Delta t H(x) - y \cdot x \tag{45}$$

in $\mathbb{R}^N$. This functional has indeed a unique minimizer $x \in \mathbb{R}$ for every $y \in \mathbb{R}^N$, since it is continuous, convex and coercive.

This minimizer is precisely the unique solution of (44), since J is also strictly convex.

The discrete iteration (41) is thus the same, written in the form (43), which would be used when looking for a fixed point of the map $J = (I + \Delta t \nabla H)^{-1}$, if J were contractive. We will now show that J is indeed contractive.

Consider two points $y_1, y_2 \in \mathbb{R}^N$ and the corresponding solutions $x_1, x_2 \in \mathbb{R}^N$:

$$x_j + \Delta t \nabla H(x_j) = y_j,\ j = 1,\, 2.$$

Taking the difference of the equations for $j = 1, 2$, we obtain:

$$x_1 - x_2 = \Delta t\big(\nabla H(x_1) - \nabla H(x_2)\big) = y_1 - y_2.$$

Taking the scalar product of this identity with $x_1 - x_2$, we obtain that:

$$\mid x_1 - x_2 \mid^2 + \Delta t \langle \nabla H(x_1) - \nabla H(x_2),\, x_1 - x_2 \rangle = \langle y_1 - y_2,\, x_1 - x_2 \rangle \leq \mid y_1 - y_2 \mid \mid x_1 - x_2 \mid .$$

Now, if H is uniformly and strictly convex, there is some $\alpha > 0$ such that:

$$\langle \nabla H(x_1) - \nabla H(x_2),\, x_1 - x_2 \rangle \geq \alpha \mid x_1 - x_2 \mid^2,$$

so we deduce that:

$$(1 + \alpha \Delta t) \mid x_1 - x_2 \mid^2 \leq \mid y_1 - y_2 \mid \mid x_1 - x_2 \mid .$$

In other words:

$$\mid x_1 - x_2 \mid \leq k \mid y_1 - y_2 \mid$$

with:

$$k = \frac{1}{1 + \alpha \Delta t} < 1.$$

Let's see then that $J = (I + \Delta t \nabla H)^{-1}$ is strictly contractive. The iteration (41) is built by discretizing in time the gradient system, and thus, it converges to the fixed point of J satisfying:

$$x + \nabla t \nabla H(x) = x.$$

This equation is obviously equivalent to:

$$\nabla H(x) = 0$$

whose unique solution is the global minimum x^* of H.

We have shown that the discrete dynamical system (41) gives an iterative algorithm for approximating the minimum of the functional H.

4.5 Least squares

We have seen that many of the problems that are posed in the context of fluid mechanics admit a variational formulation. From the theoretical point of view, this allows to solve using the direct method in the calculus of variations and, from the computational point of view, to do so using an iterative descent method as the conjugate gradient method.

There are, however, many other problems that can not be addressed in this way, either because they do not admit a variational formulation, or because the functional that is involved is not convex.

This is the case of linear systems:

$$Ax = b, \tag{46}$$

in which, if A is not symmetric, then $Ax-b$ is not the gradient of a functional $J : \mathbb{R}^N \to \mathbb{R}$.

On the other hand, in the context of reaction-diffusion equations, there appears elliptic equations with an exponential nonlinearity of the form:

$$\begin{cases} -\Delta u = e^u + f & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \partial\Omega. \end{cases}$$

In this case, the solutions are critical points of the functional:

$$J(u) = \frac{1}{2}\int_\Omega \mid \nabla u \mid^2 dx - \int_\Omega e^u dx - \int_\Omega f u dx,$$

which is well-defined if $\Omega \subseteq \mathbb{R}^2$. However, since J is not convex, the usual descent methods do not guarantee the convergence to a critical point.

In these cases, the methods inspired by least-squares can be useful.

To illustrate this type of methods, we will first consider a system of N nonlinear equations in $\mathbb{R}^N$:

$$f_i\big(x_1, \ldots, x_N\big) = 0,\ i = 1, \ldots, N, \tag{47}$$

that we can write in vector form as:

$$f(x) = 0. \tag{48}$$

Let Σ be a $N \times N$ symmetric, positive-definite matrix, and consider the function:

$$j(y) = \frac{1}{2}\langle \Sigma f(y),\, f(y)\rangle \tag{49}$$

where $< \cdot, \cdot >$ denotes the scalar product in $\mathbb{R}^N$.

The least-squares solution associated to matrix Σ is the one solving the following minimization problem:

$$\begin{cases} x \in \mathbb{R}^N \\ j(x) \leq j(y), \quad \forall y \in \mathbb{R}^N. \end{cases} \tag{50}$$

To illustrate the significance of this reduction, we will consider the linear equation:

$$f(y) = Ay - b. \tag{51}$$

The original equation that we want to solve is thus:

$$Ax = b, \tag{52}$$

which admits a solution if and only if:

$$b \in R(A) = \{q : \, q = Ay, \, y \in \mathbb{R}^N\}. \tag{53}$$

In order to simplify the computations, we will take $\Sigma = I$. In this case, the solutions obtained using least-squares are the ones corresponding to the *normal form*:

$$A^t Ax = A^t b. \tag{54}$$

The surprising thing about this is that the matrix $A^t A$ which is involved in the normal equation is symmetric and positive-semidefinite. Besides, this system admits always at least one solution, since $b \in R(A^t) = R(A^t A)$.

When $\ker A^t A = 0$ (i.e. $\mathrm{rank}(A) = N$) the system (54) admits a unique solution. However, when $\mathrm{rank}(A) < N$, the system admits infinitely many solutions of the form:

$$x = \hat{x} + z, \, \hat{x} \in R(A^t), \, z \in \ker A. \tag{55}$$

From the solutions:

$$\mathbb{R}^N = R(A^t) \oplus \ker(A); \, (R(A^t))\pm = \ker(A)), \tag{56}$$

we deduce the relation:

$$\| x \|^2 = \| \hat{x} \|^2 + \| z \|^2 \geq \| \hat{x} \|^2 . \tag{57}$$

We thus see that $\hat{x}$ is the unique solution of minimum norm of (54), and is characterized as being the critical point of the functional:

$$J(x) = \frac{1}{2} \| A^t x \|^2 - A^t b \cdot x, \tag{58}$$

which is convex. The method on least-squares plays the role of the convexifier of system (52).

Let's now consider the case then f is not affine. Suppose that $f \in C^2$. Let x be a solution to (48), then:

$$\begin{cases} j'(x) = f'(x)^t \Sigma f(x) = 0 \\ j''(x) = f'(x)^t \Sigma f'(x). \end{cases} \tag{59}$$

The matrix $j''(x)$ is positive-semidefinite and, when $f(x)$ is regular (i.e. $\det(f'(x)) \neq 0$) then it is positive-definite. In this way we observe that $j''(\cdot)$ is positive-definite in a neighborhood of x, and thus j is strictly convex. We thus see that the least-squares method has local convexification properties.

The role of the matrix Σ chosen to apply the least-squares method is multiple. For example, it allows to give more importance to some of the equations $f_i(x) = 0$ among others. The matrix Σ can also help reduce the condition number of $j'(y)$, which makes the least-squares problem (50) more robust.

In the linear setting, the equation corresponding to each matrix Σ is of the form:

$$A^t \Sigma A x = A^t \Sigma b. \tag{60}$$

This is the so-called generalized normal equation. This system admits always one solution, that may or may not be unique depending on $\ker A$ being 0 or not.

When the dimension of the system is not exceedingly large, the system (60) can be solved using a direct method (for example, Cholesky). When $\text{rank}(A^t) < N$, it is convenient to introduce a regularization of the system of the form:

$$(\varepsilon S + A^t \Sigma A) x_\varepsilon = A^t \Sigma b, \tag{61}$$

being $\varepsilon > 0$ and S a symmetric positive-definite matrix (for example $S = I$). We have then that:

$$x_\varepsilon - \hat{x}_s = O(\varepsilon), \tag{62}$$

where $\hat{x}_s$ is the unique solution to (60) in $R(S^{-1}A^t)$. On the other hand, the equation (61) can be solved using the conjugate gradient method.

5 The Burgers equation

5.1 Presentation

The viscous Burgers equation is the following:

$$u_t - \nu u_{xx} + (u^2)_x = 0. \tag{63}$$

Where $\nu > 0$ in (63) is the viscosity parameter.

In the case $nu = 0$, equation (63) turn into the following first order nonlinear hyperbolic equation:

$$u_t + (u^2)_x = 0. \tag{64}$$

In both cases, it is the $1 - d$ analogue of the Navier-Stokes and Euler equations of fluid dynamics.

The Navier-Stokes equations for a homogeneous and incompressible fluid can be written as:

$$\begin{cases} u_t - \nu \Delta u + u \cdot \nabla u = \nabla p \\ u = 0. \end{cases} \tag{65}$$

Where in (65) $u = u(x,t)$ is the velocity vector of the fluid that can thus have three components, or two in the simplified case of bi-dimensional fluids. The parameter $\nu > 0$ gives the viscosity of the fluid. The function $p = p(x,t)$ is a scalar function representing the fluid pressure.

The first equation in (65) is actually a system of three (resp. two in dimension 2) partial differential equations with three (resp two) unknowns. The second equation reflects the incompressibility of the fluid.

In absence of viscosity, i.e., when $nu = 0$, we obtain the Euler equations for an ideal fluid:

$$\begin{cases} u_t + u \cdot \nabla u = \nabla p \\ u = 0. \end{cases} \tag{66}$$

Which is indeed an idealization, since every fluid has some degree of viscosity. However, (66) is a useful model for describing the movement of low-viscous fluids.

Equations (63) and (64) are reduced models of the Navier-Stokes (65) and Euler (66) equations respectively.

Equation (63), sometimes also called non-linear heat equation, it's the easiest case in which the effects of viscosity and nonlinear quadratic convection are combined. Using a change of variables introduced independently by Hopf and Cole in the fifties, this equation can be reduced to a linear heat equation, and thus solved explicitly. In the following section we will describe this change of variables. The solution obtained in this way adequately reflects the presence of the two terms and the main effects present in (63): viscosity and nonlinear convection. Viscosity makes the solution scale as a Gaussian, while nonlinear convection develops an assymetry in the solution, which is the result of a transport phenomenon with at non-homogeneous velocity.

The solutions of (64) are the limit when $\nu \to 0$ of (63). In this case, the viscosity effect disappears, and this allows for the possibility that solutions lose their regularity in finite time, developing the so-called shocks.

5.2 The Hopf-Cole transform

With the aim of introducing this transform, we will now consider solutions $u = u(x,t)$ of (63) such that $|u(x,t)| + |u_x(x,t)| \to 0$ as $|x| \to \infty$.

If $u = u(x,t)$ is a solution of (63) of this type, then the function:

$$v = v(x,\, t) = \int_{-\infty}^{x} u(s,\, t)ds \tag{67}$$

satisfies:

$$v_t - \nu v_{xx} + \mid v_x \mid^2 = 0. \tag{68}$$

We thus define:

$$w = v(x,\, t/\nu)$$

which verifies:

$$w_t - w_{xx} + \frac{1}{\nu} \mid w_x \mid^2 = 0. \tag{69}$$

On the other hand:

$$z = 2/\nu \tag{70}$$

satisfies then:

$$z_t - z_{xx} + \mid z_x \mid^2 = 0. \tag{71}$$

We finally introduce:

$$\eta(x,\, t) = e^{-z} \tag{72}$$

satisfying the heat equation:

$$\eta_t - \eta_{xx} = 0. \tag{73}$$

Undoing this change of variables we observe that:

$$\begin{aligned} &u = v_x \\ &v(\cdot,\, t/\nu) = w(\cdot,\, t) = \nu z(\cdot,\, t) = -\nu \log(\eta). \end{aligned}$$

And thus:

$$u(x,\, t) = -\nu \frac{\eta_x(x,\, \nu t)}{\eta(x,\, \nu t)}. \tag{74}$$

The solution η of the heat equation is obtained by convolution with the Gauss kernel:

$$G(x,\, t) = (4\pi t)^{-1/2} \exp - \mid x \mid^2 4t, \tag{75}$$

so that:

$$\eta(x,\, t) = (G(\cdot,\, t) * \eta_0(\cdot))(x), \tag{76}$$

where η_0 is the initial datum of η.

On the other hand:

$$G_x(x,\, t) = -\frac{x}{4\sqrt{\pi} t^{3/2}} \exp - \mid x \mid^2 4t. \tag{77}$$

We obtain:

$$u(x,\, t) = \frac{\int_{\mathbb{R}} (x-y) e^{-|x-y|^2/4\nu t} \eta_0(y) dy}{2t \int_{\mathbb{R}} e^{-|x-y|^2/4\nu t} \eta_0(t) dy}. \tag{78}$$

Now, we have that:

$$\eta_0(x) = e^{-\int_{-\infty}^{x} u_0(\sigma) d\sigma/\nu}. \tag{79}$$

So:

$$u_\nu(x,\, t) = \frac{\int_{\mathbb{R}} (x-y) e^{-H(x,\, y,\, t)/\nu} dy}{2t \int_{\mathbb{R}} e^{-H(x,\, y,\, t)/\nu} dy} \tag{80}$$

where:

$$H(x,\, y,\, t) = \frac{|\, x - y \,|^2}{4t} + \int_{-\infty}^{y} u_0(\sigma) d\sigma. \tag{81}$$

From the expression of the solution we observe that:

• the solution function $u = u_\nu(x, t)$ is regular for every $x \in \mathbb{R}$ and $t > 0$ when u_0 is, for example, in the class $L^1(\mathbb{R})$. In order to see this, it's enough to use the regularizing effect of the convolution with the Gaus kernel deduced from Young's inequality:

$$\| f * g \|_\infty \leq \| f \|_1 \| g \|_\infty . \tag{82}$$

This inequality is applied using that $G(\cdot, t)$ and all its derivatives of arbitrary order are, for every $t > 0$, regular and integrable functions, and that the initial datum $e^{-\nu \int_{-\infty}^{x} u_0(\sigma) d\sigma}$ is bounded because $\int_{-\infty}^{x} u_0(\sigma) d\sigma$ is bounded, which is in turn a consequence of the initial datum u_0 being integrable.

• If the initial datum u_0 is an even function, the solution is not even. This is a consequence of the effect of the nonlinear convolution. This does not happen for the solutions of the linear heat equation. Indeed, the solutions of:

$$u_t - u_{xx} = 0,\, x \in \mathbb{R}, t > 0 \, u(x,\, 0) = u_0(x),\, x \in \mathbb{R}$$

are of the form:

$$u(x,\, t) = [G(\cdot,\, t) * u_0(\cdot)](x)$$

Is thus easy to see that u is even (resp. odd) with respect to x for every $t > 0$, depending of the parity of u_0.

This, however, is no loner true for the solutions of the Burgers equation that are given by (70). This is in part due to the fact that, in (63) the initial datum $e^{-\nu \int_{-\infty}^{x} u_0(\sigma) d\sigma}$ has a contribution that does not preserve the parity of the initial datum, and to the fact that in that expression not only the kernel G intervenes, but also G_x.

Everything that we have said until now is valid when $\nu > 0$. The expression (76) has a singularity for $\nu = 0$. As we will see below, the limit when $\nu \to 0$ of the solution $u_\nu ? u_\nu(x, t)$ of (63) is the solution of the inviscid Burgers equation (66).

Let's now analyze (64). In this case, the Hopf-Cole transform doesn't apply. The only idea that can be applied from the viscous case is the change of variables:

$$v(x,\, t) = \int_{-\infty}^{x} u(\sigma,\, t) d\sigma$$

which reduces (64) to the Hamilton-Jacobi equation:

$$v_t + |\, v_x \,|^2 = 0.$$

However, the Hamilton-Jacobi equation cannot be linearized, so it's better to directly solve (64) using the method of characteristics.

We write (64) as:

$$u_t + 2uu_x = 0. \tag{83}$$

This allows to chech that the solutions of (82), while being of class C^1, are constant along the characteristic curves, i.e.:

$$u(x(t),\, t) = C$$

where $x = x(t)$ is characterized by the equation:

$$x'(t) = 2u(x(t),\, t). \tag{84}$$

This curves are easy to compute. Indeed, as u is constant along characteristics, $u(x(t), t)$ has to coincide with its value at $t = 0$, so that:

$$u(x(t),\, t) = u_0(x_0), \tag{85}$$

where x_0 is the initial point of the characteristic curve. The equation of the characteristic line is then:

$$x(t) = 2u_0(x_0)t + x_0 \tag{86}$$

and thus:

$$u(x,\, t) = u_0(x_0), \tag{87}$$

where (x, t) and x_0 are related via the identity (86).

In virtue of this analysis, we can check that u is constant along characteristic lines of slope $1/2u_0$ in the plane (x, t). From this analysis, it is deduced that if the initial datum u_0 is decreasing, then u will produce a discontinuity in finite time. Indeed, since there are two points x_0, x_1 such that $x_0 < x_1$ and $u(x_0) > u(x_1)$, the characteristic lines starting from x_0 and x_1 will meet in a finite time t^* at a point x^*. The solution will thus be not continuous at (x^*, t^*), since the two values $u_0(x_0)$ and $u_0(x_1)$ are incompatible.

A more careful analysis allows to compute the instant t^* in which the discontinuity of shock is produced. Indeed, the characteristics that, from equation (85), start from x_0 and x_1, will meet at time t^* if:

$$2u_0(x_0)t + x_0 = 2u_0(x_1)t + x_1.$$

This happens exactly at time:

$$t^* = \frac{x_1 - x_0}{2(u_0(x_0) - u_0(x_1))} = -\frac{x_0 - x_1}{2(u_0(x_0) - u_0(x_1))}. \tag{88}$$

When $x_0 \to x_1$ the instant of time t^* takes the limit value:

$$t^* = -\frac{1}{2u_0'(x_0)}. \tag{89}$$

From this expression we deduce that the minimum time in which the shock occurs is:

$$t^* = \frac{1}{2\max_{x_0\in\mathbb{R}}(-u_0'(x_0))}. \tag{90}$$

We thus see that the behavior of the viscous Burgers equation (63) and the inviscid Burgers equation (64) are very different from each other. While the solutions of (63) are regular for every $\nu > 0$, the solutions of (64) are discontinuous when the initial datum is decreasing. In fact, it's enough for the initial datum to be decreasing in some interval for the shocks to occur in finite time. In spite of this, as we will see in the following section, the solutions of the inviscid Burgers equation are the limit of the solutions of the Burgers equation as $\nu \to 0^*$.

5.3 Vanishing viscosity

In this section we analyze the limiting behavior as $\nu \to 0$ of the solutions (80) of the Burgers equation (63).

When taking the limit in (80) as $\nu \to 0^+$, the integrals that intervene are concentrated aroung the points in which H attains its minimum. We thus compute the critical points of H:

$$H_y = -\frac{x-\xi}{2t} + u_0(y) = 0 \Leftrightarrow \xi = x - 2tu_0(y), \tag{91}$$

in which:

$$H = -tu_0^2(\xi) + \int_{-\infty}^{\xi} u_0(\sigma)d\sigma. \tag{92}$$

The contribution of an integral:

$$\int_{\mathbb{R}} f(y)e^{-H/\nu}dy \tag{93}$$

in a neighborhood of the minimum $y = \xi$ is:

$$f(\xi)\sqrt{\frac{2\pi\nu}{H''(\xi)}}e^{-H(\xi)/\nu}. \tag{94}$$

In our case:

$$H''(\xi) = \frac{1}{2t}. \tag{95}$$

Appplying these formulas in the integrals intervening in (80), we obtain:

$$\int_{\mathbb{R}} (x-y)e^{-H/\nu}dy \sim (x-\xi)\sqrt{\frac{\pi\nu}{t}}e^{-[tu_0^2(\xi)+\int_{-\infty}^{\xi} u_0(\sigma)d\sigma]\nu}, \tag{96}$$

$$\int_{\mathbb{R}} e^{-H/\nu}dy \sim \sqrt{\frac{\pi\nu}{t}}e^{-[tu_0^2(\xi)+\int_{-\infty}^{\xi} u_0(\sigma)d\sigma]}. \tag{97}$$

And thus:

$$u_\nu(x,\, t) \sim \frac{(x-\xi)}{2t} \tag{98}$$

where ξ is characterized by the equation:

$$\xi = x - 2tu_0(\xi) \tag{99}$$

which is exactly the expression obtained in the previous section for the inviscid Burgers equation (64), since $(x-\xi)/2t = u_0(\xi)$.

This expression holds when the function H has only one minimum. In case H has several minima $\xi_1, \ldots, \xi_N$, each one of them contributes in a similar way to the integrals appearing in (80). However, due to the presence of the exponential factor, in the determination of the asymptotic expression of u_ν only the global minima of H intervene. If, for example, H only admits two global minima ξ_1, ξ_2, then the asymptotic expression of u_ν would be:

$$u_\nu(x,\, t) \sim u_0(\xi_1) + u_0(\xi_2). \tag{100}$$

Below we will rigorously justify this asymptotic expression. However, for the moment let's just analyze its significance in regard to the Burgers equation (64).

We will distinguish two cases:

Increasing initial datum u_0

In this case, as we saw in the previous section, the method of characteristics does not predict any shock, and we expect the Burgers equation to admit a regular solution as long as the initial datum u_0 is regular.

The asymptotic analysis that we have just carried out confirms this fact. Indeed:

$$u_\nu(x,\, t) \to u_0(\xi),\ \nu \to 0, \tag{101}$$

where $\xi \in \mathbb{R}$ is characterized by the equation:

$$\xi + 2tu_0(\xi) = x. \tag{102}$$

For $t > 0$, is $u_0(\cdot)$ is increasing, the function:

$$\xi \to \xi + 2tu_0(\xi), \tag{103}$$

is also increasing, and thus (102) admits a unique solution. This confirmes without any ambiguity that the limit of the solutions u_ν of the viscous Burgers equation, as $\nu \to 0$, is the solution of the inviscid Burgers equation obtained via the method of characteristics.

Let's now consider the particular case of a discontinuous, piecewise constant initial datum:

$$u_0(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases} \tag{104}$$

We will solve the inviscid Burgers equation (64) with this initial datum. This is known as a Riemann problem, which is a basic element of the known Riemann method for the approximation of the initial datum using piecewise constant initial datum.

If we apply the method of characteristics with the initial datum u_0 of (104), we obtain that the solution u of (64) is of the form:

$$u = \begin{cases} 0, & x < 0,\, t > 0 \\ 1, & x \geq t. \end{cases} \tag{105}$$

However, the method of characteristics does not yield any value for the solution u in the region $0 < x < t$.

The vanishing viscosity method is actually a way to obtain a globally defined solution in this case. Indeed, let's consider equation (102) in this particular case. The system is written as:

$$\begin{cases} \xi = x, & \text{si} \quad \xi < 0 \\ \xi + 2t\xi = x, & \text{si} \quad \xi > 0. \end{cases} \tag{106}$$

When $x < 0$, this yields a solution $\xi = x$ and thus, according to (101), the limit value $u = u_0(\xi) = 0$. This coincides with what we have obtained in (105). When $x > 0$, we obtain $\xi = x/(1 + 2t)$. In the limit as $\nu \to 0$, using (102) we thus obtain:

$$u(x,\, t) = \frac{x - \xi}{t} = \frac{x}{t} \tag{107}$$

which is now globally defined for every $x \in \mathbb{R}$ and every $t > 0$.

Obviously, the front $u = x/t$ connects adequately the constant value $u = 0$ at the left and the value $x = 1$ at the right. This is called a *rarefaction wave*.

We have just seen that the vanishing viscosity method gives, in the limit, a solution of the inviscid Burgers equation (64). This is the so-called *entropy solution*.

As we will see, the Burgers equation can even have several weak solutions. In this case, the entropy solution is the one that has a physical significance, since the inviscid model has to be understood as an idealization of the case in which the viscosity ν is small and tends to 0.

In the present situation there also exist other weak solutions although, as we have said, the only solution with a physical significance is the entropy solution that we have just obtained. In view of the fact that this solution u is zero for $x < 0$ and takes the value $u = 1$ for $x > t$, it is natural to consider also solutions of the form:

$$u = \begin{cases} 0, & x < \alpha t \\ 1, & x > \alpha t \end{cases} \tag{108}$$

where the line $x = \alpha t$ is to be determined. Let's now see which is the value of α that we have to choose so that the function u given by (104) is a weak solution of the Burgers equation. Recall that, since it is a solution of the Burgers equation:

$$u_t + (u^2)_x = 0 \tag{109}$$

it is necessary that:

$$\int_0^\infty \int_{\mathbb{R}} \left(u\varphi_t + u^2\varphi_x\right) dxdt = 0 \tag{110}$$

for every test function $\varphi \in C_0^\infty(\mathbb{R} \times (0, \infty))$.

In the case of a function of the form (104), (106) reduces to:

$$\int_0^\infty \int_{\alpha t}^\infty (\varphi_t + \varphi_x)\, dxdt = 0 \tag{111}$$

which occurs if and only if $\alpha = 1$.

We thus see that the Burgers equation admits also as a weak solution the solution that propagates the shock with speed 1. Now, this last shock solution is now an entropy solution. As we have seen before, the unique entropy solution is the one that develops the rarefaction wave.

A classical and important result, due to Kruzkov (see section 11.4.3 of [11]) ensures that the entropy solution is unique. This entropy solution can be characterized not only as the solution obtained by the vanishing viscosity method. Another way of characterizing it is, for example, the following unilateral bound on the derivative of the solution:

$$u_x \leq 1/2t. \tag{112}$$

This bound is obtained in the following way. Formally, if u solves (109), then $v = u_x$ satisfies:

$$v_t + (2uv)_x = v_t + 2v^2 + 2uv_x = 0. \tag{113}$$

Applying the maximum principle, we deduce that:

$$v \leq w \tag{114}$$

where $w = w(t)$ is the solution to:

$$w_t + 2w^2 = 0 \tag{115}$$

with initial datum $w(0) = \infty$. This solution can be explicitely computed: $w(t) = 1/2t$.

Now, can we justify the use of the maximum principle for equations of the form (113) to obtain the comparison (113)? This justification can be indeed carried out for entropy solutions. Indeed, if u is an entropy solution, then it is the limit as $\nu \to 0$ of solutions u_ν with viscosity $\nu > 0$. Differentiating, we see that $v_\nu = u_{\nu,x}$ satisfies:

$$v_{\nu,\,t} - \nu v_{v,\,xx} + 2v_\nu^2 + 2u_\nu v_{\nu,\,x} = 0. \tag{116}$$

We can apply the maximum principle in this parabolic equation to deduce:

$$v_\nu \leq 1/2t \tag{117}$$

for every $\nu > 0$. Taking the limit as $\nu \to 0$, we obtain the bound (112) for $u_x = v$.

Decreasing initial datum u_0

From the previous analysis, we have to study the values of ξ for which:

$$\xi + 2tu_0(\xi) = x. \tag{118}$$

Now, since u_0 is decreasing, it is not guaranteed that (118) admits a unique solution for every $x \in \mathbb{R}$ and $t > 0$. It's actually the ocntraty: for $t > 0$ sufficiently large, the term $2tu_0(\cdot)$ at the left hand side of (118) will dominate and destroy the monotonic behavior of the function $\xi + 2tu_0(\xi)$. In fact, the analysis of the characteristics already predicted the appearance of shocks in this case. Both facts are a consequence of the same phenomenon.

We will again consider the case of the Riemann problem with initial datum:

$$u_0(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0. \end{cases} \tag{119}$$

In this case, the change of variables (118) cannot be applied because u_0 is not integrable at $-\infty$. To avoid this difficulty, we define:

$$\tilde{u}_\nu(x, t) = -u_\nu(-x, t) \tag{120}$$

so that:

$$u_\nu(x, t) = -\tilde{u}_\nu(-x, t). \tag{121}$$

The function $\tilde{u}$ is a solution to the same viscous Burgers equation, but with the different initial datum:

$$\tilde{u}_0 = \begin{cases} 0, & x < 0 \\ -1, & x > 0. \end{cases} \tag{122}$$

We have now to study the solution of the equation:

$$\xi + 2t\tilde{u}_0(\xi) = x \tag{123}$$

which can be written in the form:

$$\begin{cases} \xi = x, & \text{if} \quad \xi < 0 \\ \xi - 2t = x, & \text{if} \quad \xi > 0. \end{cases} \tag{124}$$

From (124) we deduce that por $-2t < x < 0$ two distant solutions $\xi_1 = x$ and $\xi_2 = x + 2t$ are obtained. For the rest of the values (x, t), a unique solution is obtained.

We now have to analyze in which of these critical points ξ the value of H is minimum. According to (92), in each critical point the value of H is:

$$H(\xi) = t\tilde{u}_0(\xi) + \int_{-\infty}^{\xi} \tilde{u}_0(s)ds. \tag{125}$$

And thus, when $\xi < 0$:

$$H(\xi) = 0 \tag{126}$$

meanwhile, when $\xi > 0$, for example, if $\xi = \xi_2$,

$$H(\xi_2) = t - \int_0^{\xi_2} ds = t - \xi_2 = t - x - 2t = -(t + x). \tag{127}$$

And thus:

$$\begin{cases} H(\xi_1) < H(\xi_2) & \text{if} \quad t + x < 0 \\ H(\xi_2) < H(\xi_1) & \text{if} \quad t + x > 0. \end{cases} \tag{128}$$

We thus deduce that:

$$u_\nu(x,\, t) \sim u(x,\, t) \tag{129}$$

where

$$u(x,\, t) = \begin{cases} 1, & x < t \\ 0, & x > t. \end{cases} \tag{130}$$

Again, int he limit (130) we obtain an entropy solution of the inviscid Burgers equation (64). It is a shock wave that propagates with speed 1. This is coherent with the Rankine-Hogoniot condition, which is needed for a discontinuous function to be a weak solution of the Burgers equation. Assuming that $u^\pm$ are the left and right values of the shock, then the speed of propagation is precisely:

$$s = \frac{(u^+)^2 - (u^-)^2}{u^+ - u^-}.$$

In our case, with values $u^- = 1$ y $u^+ = 0$, this gives a speed of propagation of 1.

We have thus seen that the method of vanishing viscosity allows to obtain the entropy solution of the Burgers equation. Although we have not proven here the uniqueness of the entropy solution, this is the unique solution which is physically meaningful.

While the solution is regular, it coincides with the solution obtained using the method of characteristics. When it is not, we have two possibilities: it is either a rarefaction wave, or a shock wave, depending of the sign of the shock, i.e., of the relative values of the solution at each side of the shock. Then a shock occurs, this propagates with the speed given by the Rankine-Hugoniot condition.

With the aim of completing this section, we will lastly check that the asymptotic value of the integral:

$$\int_{\mathbb{R}} f(y) e^{-H/\nu} dy$$

as $\nu \to 0$ is of the order of:

$$f(\xi) \sqrt{\frac{2\pi\nu}{H''(\xi)}} e^{-H(\xi)/\nu},$$

being ξ the unique minumum of H. Rigorously, this means that:

$$\lim_{\nu\to\infty} \frac{\int_{\mathbb{R}} f(y)e^{-H(y)/\nu}dy}{f(\xi)\sqrt{\frac{2\pi\nu}{H''(xi)}}e^{-H(\xi)/\nu}} = 1.$$

In order to prove this, it's enough to see that:

$$J_\nu(y) = \left(\frac{2\pi\nu}{H''(\xi)}\right)^{-1/2} e^{-(H(y)-H(\xi))/\nu}$$

is an approximation of the identity as $\nu \to \infty$. It is obvious that, as $\nu \to \infty$, then $J_\nu(y) \to 0$ exponentially except at the minimum point $y = \xi$.

On the other hand:

$$\int_{\mathbb{R}} J_\nu(y)dy \to 1,\ \nu \to \infty.$$

Indeed:

$$H(y) - H(\xi) = \frac{H''(\xi)}{2}(y\xi)^2 + O\left(\mid y-\xi \mid^3\right)$$

and thus making the change of variables:

$$\sigma = \sqrt{\frac{H''(\xi)}{2\nu}}(y-\xi)$$

the problem reduces to check that:

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\eta^2}e^{-0(\nu^{1/2}\eta^3)}d\eta \to 1$$

which can be proven using the dominated convergence theorem.

If the reader is interested in a more detailed study of hyperbolic conservation laws, the book of Evans [11] is a good reference.

6 Splitting

6.1 Introduction

In this section, we will briefly analyze the so-called splitting methods for the numerical solution of the viscous Burgers equation:

$$u_t - \nu u_{xx} + \left(u^2\right)_x = 0. \tag{131}$$

Throughout this section, we will suppose that the viscosity $\nu > 0$ is fixed.

Many of the ideas that we will introduce here will also be useful in the more realistic and complex context of the viscous Navier-Stokes equations.

In view of the structure of equation (131), it is obvious that two models are present. On the one hand, we have the heat equation:

$$u_t - \nu u_{xx} = 0 \tag{132}$$

and, on the other hand, the inviscid Burgers equation:

$$u_t + \left(u^2\right)_x = 0. \tag{133}$$

Indeed, using the Hopf-Cole transform, in the previous section we showed that the solution to (131) had both viscous and convective effects.

In this type of situations in which the model in consideration has two recognisble subsystems, it is natural to use decomposition or "splitting" methods, that allow to obtain the solution to the global system from the solution of each subsystem and, on the other hand, to use a different numerical method for each of the subsystems under consideration.

Lie's theorem shows that for any pair of $n \times n$ matrices A_1 and A_2, the following formula holds:

$$e^{A_1+A_2} = \lim_{j\to\infty} \left(e^{A_1/j} e^{A_2/j}\right)^j, \tag{134}$$

whether the matrices A_1 and A_2 commute or not. This is useful when solving a differential equation of the form:

$$x' = \Big(A_1 + A_2\Big)x,\ t \in \mathbb{R},\ x(0) = x_0, \tag{135}$$

since its solution is given by:

$$x(t) = e^{(A_1+A_2)t} x_0. \tag{136}$$

The viscous Burgers equation (131) can also be understood as a problem of the form (135) although, in this case, the unknown $u = u(x, t)$ is, for any $t > 0$, a function depending of x, and thus belonging to a infinite dimensional space. A_1 is the differential operator $A_1 u = \partial_x^2 u$ and A_2 is the non-linear operator $A_2 u = -(u^2)_x$.

In this section we will introduce splitting methods for the discrete version of these equations.

We will thus consider the model (135), that can be discretized in time with step Δt substituting, as sual, the derivative $x'(t)$ by a difference quotient in the interval $[n\Delta t, (n+1)\Delta t]$. The type of splitting schemees that we will use are based in the use of an additional node (or several), for example in the mid point $(n + 1/2)\Delta t$, so that in each subinterval $[n\Delta t, (n + 1/2)\Delta t]$ and $[(n + 1/2)\Delta t, (n + 1)\Delta t]$ we can see (135) as a system that is dominated either by A_1 or by A_2.

In the section below we will analyze the validity of the time discretization when approximating solutions of evolution PDEs. As we will see, this type of methods, that are classical and well understood in the contest of ODEs, can also be applied to PDEs, even in the non-linear setting in which the classical theory of linear semigroups is not applicable, where specific ideas from non-linear semigroup theory become necessary.

6.2 Peaceman-Rachford

We will first introduce the Peaceman-Rachford scheme:

$$\begin{cases} x^0 = x^0 \\ \frac{x^{n+1/2}-x^n}{\Delta t/2} = A_1 x^{n+1/2} + A_2 x^n \\ \frac{x^{n+1}-x^{n+1/2}}{\Delta t/2} = A_1 x^{n+1/2} + A_2 x^{n+1}. \end{cases} \tag{137}$$

We see that the resulting system is doubly implicit, first when passing from x^n to $x^{n+1/2}$ and then when passing from $x^{n+1/2}$ to x^{n+1}. In each case, however, one of the operators A_1 or A_2 is playing the main role, and the other one is seen as a perturbation that incorporates information from the previous step.

In order to understand the effect that this splitting scheme has sover a system, we will start by considering the case of the easiest possible system:

$$x'(t) = Ax(t),\ t > 0;\ x(0) = x_0, \tag{138}$$

where A is a symmetric, negative-definite matrix for which the solution is of the form:

$$x(t) = e^{At} x_0. \tag{139}$$

If $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of the matrix A and $e_1, \ldots, e_N$ are the corresponding eigenvectors, then the solutions are linear combinations of elementary solutions of the form:

$$x(t) = e^{\lambda_j t} e_j,\ j = 1, \cdots, N. \tag{140}$$

As $\lambda_j \leq 0$ for ever $j = 1, \ldots, N$, it's easy to see that:

$$\mid x(t) \mid \leq \mid x_0 \mid,\ \forall t \geq 0, \tag{141}$$

for every solution.

We will now decompose the matrix A as:

$$A = \alpha A + \beta A \tag{142}$$

so that:

$$A_1 = \alpha A,\ A_2 = \beta A, \tag{143}$$

with:

$$\alpha > 0,\ \beta > 0,\ \alpha + \beta = 1. \tag{144}$$

Applying the splitting scheme (137), we obtain:

$$x^{k+1} = \left(I - \frac{\Delta t}{2}\beta A\right)^{-1} \left(I + \frac{\Delta t}{2}\alpha A\right) \left(I - \frac{\Delta t}{2}\alpha A\right)^{-1} \left(I + \frac{\Delta t}{2}\beta A\right) x^k.$$

Iterating, we obtain:

$$x^k = \left(I - \frac{\Delta t}{2}\beta A\right)^{-k} \left(I + \frac{\Delta t}{2}\alpha A\right)^k \left(I - \frac{\Delta t}{2}\alpha A\right)^{-k} \left(I + \frac{\Delta t}{2}\beta A\right)^k x^0.$$

When we apply the previous identity in the direction of each of the eigenvectors e_j, i.e., with $x^0 = e_j$, we deduce that:

$$x_j^k = \left(\frac{1 + \frac{\Delta t}{2}\alpha\lambda_j}{1 - \frac{\Delta t}{2}\alpha\lambda_j}\right)^k \left(\frac{1 + \frac{\Delta t}{2}\beta\lambda_j}{1 - \frac{\Delta t}{2}\beta\lambda_j}\right)^k e_j.$$

Now, noting that:

$$\left|\frac{1+\xi}{1-\xi}\right| \leq 0,\ \forall \xi < 0$$

we obtain the stability of the splitting scheme, since:

$$\left|x_j^k\right| \leq \mid e_j \mid,\ \forall k \geq 0,\ \forall j = 1, \cdots, N.$$

Besides, in the case in which $\lambda_j < 0$ for every $j = 1, \ldots, N$, in which $x(t) \to 0$ when $t \to \infty$ exponentially, the discrete solution also reproduces this behavior, since:

$$\left|\frac{1 + \frac{\Delta t}{2}\alpha\lambda_j}{1 - \frac{\Delta t}{2}\lambda_j}\right| < 1,\ \left|\frac{1 + \frac{\Delta t}{2}\beta\lambda_j}{1 - \frac{\Delta t}{2}\beta\lambda_j}\right| < 1,\ j = 1, \cdots, N.$$

In order to study this scheme in a more detailed way, we introduce the rational function:

$$R_1(\xi) = \left(\frac{1 + \frac{\alpha\xi}{2}}{1 - \frac{\alpha\xi}{2}}\right) \left(\frac{1 + \frac{\beta\xi}{2}}{1 - \frac{\beta\xi}{2}}\right)$$

which admits, in a neighborhood of $\xi = 0$, the Taylor expansion:

$$R_1(\xi) = 1 + \xi + \frac{\xi^2}{2} + \left(\alpha^2 + \beta^2 + \alpha\beta\right)\frac{\xi^3}{4} + O(1)\xi^4$$

while the exponnetial function that is involved in the solution of the differential equation has the expansion:

$$e^\xi = 1 + \xi + \frac{\xi^2}{2} + \frac{\xi^3}{6} + O(\xi^4).$$

From these two Taylor expansions we deduce that the scheme is consistent of order two for any choice of the parameters $0 < \alpha, \beta < 1$ so that $\alpha + \beta = 1$. The smallest difference in the third order term is produced for $\alpha = \beta = 1/2$.

The scheme is thus convergent of order two.

The only inconvenient of this method is its lack of absolute stability or stiff stability, which is manifested in the fact that $R_1(\xi) \to 1$ as $\xi \to -\infty$. As ξ plays the role of $\lambda \Delta t$, this shows that the exponential rate of convergence is lost as $|\Delta t|$ increases. But this is unavoidable in the context of parabolic equations, in which typically λ tends to $-\infty$. Whatever value of $\Delta t > 0$ is chosen, no matter how small, the numerical scheme is not capable of reproducing the dynamics of the PDE as $|\lambda|$ increases, and thus the exponential stability of the solutions is emphasized.

The Peaceman-Rachform method that we have just described is only one instance of the family of splitting or alternate direction method whose scope of application is very wide.

Let's for example consider the Dirichlet problem for the heat equation:

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega,\, t > 0 \\ u = 0 & \text{at} \quad \partial\Omega,\, t > 0 \\ u(x,\, 0) = u_0(x) & \text{in} \quad \Omega. \end{cases} \tag{145}$$

Let's analyze the case $\Omega \subset \mathbb{R}^2$.

The most elementary semi-discrete scheme for the numerical approximation of this system is obtained by using the five-point approximation scheme for the Laplacian. We thus obtain:

$$\begin{cases} u'_{i,j} + \frac{2u_{i,j} - u_{i+1,j} - u_{i-1,j}}{h_1^2} + \frac{2u_{i,j} - u_{i,j+1} - u_{i,j-1}}{h_2^2} = 0, & (i,j) \in \Omega, t > 0 \\ u_{i,j} = 0, & (i,j) \in \partial\Omega,\, t > 0 \\ u_{i,j}(0) = u_{0,i,j}, & (i,j) \in \Omega. \end{cases} \tag{146}$$

In this expression we use the usual notations, so thhat h_1 denotes the width of the mesh in the variable x_1, h_2 denotes the width in the variable x_2, $u_{i,j}$ the approximation of the solution u at the point $(h_1 i h_2 j) \in \Omega$.

Denoting by U_h the unknown vector $(u_{i,j})$ that contains all the numerical values, system (146) can be written as:

$$\begin{cases} \frac{dU_h}{dt} + A_{1h}U_h + A_{2h}U_h = 0,\, t > 0 \\ U_h(0) = U_{h,0}, \end{cases} \tag{147}$$

where A_{1h} y A_{2h} are the discrete analogues of $\partial^2/\partial x_1^2$ y $\partial^2/\partial x_2^2$, respectively.

This system can be approximated using the classical time discretization schemes as the explicit or implicit Euler schemes, or the Crank-Nicolson scheme. However, we can also apply the Peaceman-Rachford scheme that can be written as:

$$\begin{cases} \frac{U_h^{n+1/2} - U_h^n}{\Delta t/2} = A_{1h}U^{n+1/2} + A_{2h}U^n \\ \frac{U_h^{n+1} - U_h^{n+1/2}}{\Delta t/2} = A_{1h}U_h^{n+1/2} + A_{2h}U_h^{n+1}. \end{cases} \tag{148}$$

System (148) consists essentially in two uncoupled systems, similar to the ones obtained when discretizing the $1-d$ heat equation. Indeed, both matrices A_{1h} and A_{2h} that intervene in each of them are the usual matrices that appear in the numerical discretization of the one-dimensional heat equation. These are, thus, tri-diagonal systems that are easy to solve.

This method is called method of *alternate directions* since the $2-d$ heat equation is approximated by iteratively and alternatively solving one-dimensional heat equations in the variables x_1 and x_2, respectively.

Let's however go back to the abstract general system (138) and to its discretization (137). In (135), the operators A_1 and A_2 play completely symmetric roles, but they can also be used asymetrically. For example, if we apply (137) to the solution of (138) with $A_1 = A$ and $A_2 = 0$ we obtain:

$$\frac{x^{n+1/2}-x^n}{\Delta t/2} = Ax^{n+1/2}$$
$$\frac{x^{n+1}-x^{n+1/2}}{\Delta t/2} = Ax^{n+1/2}.$$

We thus observethat $x^{n+1/2} = \left(x^{n+1}+x^n\right)\Big/2$, which in turn implies that:

$$\frac{x^{n+1}-x^n}{\Delta t} = A\left(\frac{x^{n+1}+x^n}{2}\right).$$

Taking $A_1 = 0$ and $A_2 = A$ in the decomposition, we obtain exactly the same expression. This is, however, only true when the system under consideration is an autonomous system. Indeed, when the operator A also depends of t, while the first equation points to the system:

$$\frac{x^{n+1}-x^n}{\Delta t} = A\left(\left(n+\frac{1}{2}\right)\Delta t\right)\left(\frac{x^{n+1}+x^n}{2}\right) \tag{149}$$

the second equation gives:

$$\frac{x^{n+1}-x^n}{\Delta t} = \frac{1}{2}\left[A(nt)x^n + A((n+1)\Delta t)x^{n+1}\right]. \tag{150}$$

A more careful analysis points that, for both schemes to coincide, is not only necessary for A to be independent of t, i.e., for the system to be autonomous, but it's also necessary for A to be linear. In the case in which A is non-linear, the difference between (149) and (150) is obvious.

Both are schemes of Crank-Nicolson type, and both are of order two when the operator A is sufficiently regular in t and x.

These schemes can also be written as:

$$\begin{cases} x^{n+1/2} & = & x^n + \frac{\Delta t}{2}A\left(x^{n+1/2},\,(n+1/2)\Delta t\right) \\ x^{n+1} & = & x^n + \Delta t A\left(x^{n+1/2},\,\left(n+\frac{1}{2}\right)\Delta t\right) \\ & = & x^n + 2\left(x^{n+1/2}-x^n\right) \end{cases} \tag{151}$$

and:

$$x^{n+1} = x^n + \frac{\Delta t}{2}\left[A\left(x^n,\, n\Delta t\right) + A\left(x^{n+1},\, (n+1)\Delta t\right)\right], \tag{152}$$

respectively. These are, thus, semi-implicit Runge-Kutta schemes of order 2.

6.3 Douglas-Rachford

There exist other variants of the splitting scheme that we have considered. For example, we have the Douglas-Rachford scheme that, when x^n is known, computes the value of $\widehat{x}^{n+1}$ and x^{n+1}, being x^{n+1} the approximation of x in the next instant of time and $\widehat{x}^{n+1}$ an auxiliary value. The scheme has the form:

$$\left\{\begin{array}{lcl} \frac{\hat{x}^{n+1}-x^n}{\Delta t} & = & A_1\left(\hat{x}^{n+1},\, (n+1)\Delta t\right) + A_2\left(x^n,\, n\Delta t\right) \\ \frac{x^{n+1}-x^n}{\Delta t} & = & A_1\left(\hat{x}^{n+1},\, (n+1)\Delta t\right) + A_2\left(x^{n+1},\, (n+1)\Delta t\right). \end{array}\right. \tag{153}$$

The convergence of this scheme is known in a very general context of monotonic operators A_1 and A_2. Let's analyze, as in the previous case, the convergence in the easier case in which A is a symmetric $N \times N$ matrix.

In thic case we obtain, using the decomposition (142):

$$x^{n+1} = (I - \beta\Delta t A)^{-1}(I - \alpha\Delta t A)^{-1}(I - \alpha\beta \mid \Delta t \mid^2 A^2)x^n,$$

or, in other words:

$$x^n = (I - \beta\Delta t A)^{-n}(I - \alpha\Delta t A)^{-n}(I - \alpha\beta \mid \Delta t \mid^2 A^2)x^0.$$

Applying the scheme in each one of the eigendirections of the matrix A we deduce that:

$$x_j^k = \frac{\left(1 + \alpha\beta \mid \Delta t \mid^2 \lambda_j^2\right)^k}{(1 + \alpha\Delta t\lambda_j)^k(1 + \beta\Delta t\lambda_j)^k} x_{0j},\; j = 1, \cdots, N,\; k \geq 1.$$

The rational function associated to the scheme is thus:

$$R_2(\xi) = \frac{1 + \alpha\beta\xi^2}{(1 - \alpha\xi)(1 - \beta\xi)}.$$

As $0 < R_2(\xi) < 1$ for every $\xi < 0$, we deduce that:

$$\left|x_j^k\right| \leq \left|x_j^0\right|,\, \forall j = 1, \cdots, N,\; k \geq 1,$$

which implies the unconditional stability of the scheme. However, as:

$$R_2(\xi) = 1 + \xi + \xi^2 + O(\xi^3)$$

we observe that this Douglas-Rachford scheme is only consistent of order 1.

Again, we have that:

$$R_2(\xi) \to 1,\, \xi \to -\infty$$

which is a signal that the scheme will have a bad behavior when considering "stiff" systems with respect to absolute stability.

It is also convenient to observe that, in this scheme, the roles played by A_1 and A_2 are not symmetric.

This scheme is easier to the generalize to the splitting case for a greater number of operators than the Peaceman-Rachford scheme. If we take $A_1 = 0$ and $A_2 = A$ or $A_1 = A$ and $A_2 = 0$ in (153), in both cases we obtain the implicit Euler scheme.

6.4 θ-method

We will finish by considering the θ-method introduced by Glowinski. If we suppose that x^k is known, we compute $x^{k+\theta}$, $x^{k+1-\theta}$ and x^{k+1} in the following way:

$$\left\{\begin{array}{lcl} \frac{x^{k+\theta}-x^k}{\theta\Delta t} & = & A_1\left(x^{k+\theta},\,(k+\theta)\Delta t\right) + A_2(x^k,\,k\Delta t), \\ \frac{x^{k+1-\theta}-x^{k+\theta}}{(1-2\theta)\Delta t} & = & A_1\left(x^{k+\theta},\,(k+\theta)\Delta t\right) + A_2\left(x^{k+1-\theta},\,(k+1-\theta)\Delta t\right), \\ \frac{x^{k+1}-x^{k+1-\theta}}{\theta\Delta t} & = & A_1\left(x^{k+1},\,(n+1)\Delta t\right) + A_2\left(x^{k+1-\theta},\,(k+1-\theta)\Delta t\right). \end{array}\right. \tag{154}$$

It is convenient to distinguish this scheme from the usual θ-scheme, which is between the explicit and implicit Euler schemes.

We will analyze the scheme (154) in the case in which A is a symmetric matrix. In this case, we have:

$$x^{k+1} = (I - \alpha\theta\Delta tA)^{-2}(I + \beta\theta\Delta tA)^2(I - \beta\theta'\Delta tA)^{-1}(I + \alpha\theta'\Delta tA)x^k, \tag{155}$$

where $\theta' = 1 - 2\theta$, so that:

$$x_j^k = \frac{(1+\beta\theta\Delta t\lambda_j)^{2k}(1+\alpha\theta'\Delta t\lambda_j)^k}{(1-\alpha\theta\Delta t\lambda_j)^{2k}(1-\beta\theta'\Delta t\lambda_j)^k} e_j. \tag{156}$$

The rational function that characterized the scheme in this case is thus:

$$R_3(\xi) = \frac{(1+\beta\theta\xi)^2(1+\alpha\theta'\xi)}{(1-\alpha\theta\xi)^2(1-\beta\theta'\xi)}. \tag{157}$$

As:

$$\lim_{\xi\to-\infty} R_\xi(3) = \beta/\alpha, \tag{158}$$

in order to obtain the stability of the scheme it is necessary to impose the condition $\alpha \geq \beta$, and the condition $\alpha > \beta$ guarantees absolute stability.

The unconditional stability of the scheme demands:

$$|R_3(\xi)| \leq 1, \, \forall \xi \in \mathbb{R}^-. \tag{159}$$

A more caregul analysis shows that:

$$|R_3(\xi)| < 1, \, \forall \xi \in \mathbb{R}^- \tag{160}$$

is, at least, verified whenever θ, α and β are in the range:

$$\theta \in [1/4, \, 1/2), \, 0 < \beta < \alpha < 1, \, \alpha + \beta = 1. \tag{161}$$

On the other hand:

$$R_3(\xi) = 1 - \xi + \frac{\xi^2}{2}\left[1 + (\beta - \alpha)\left(2\theta^2 - 4\theta + 1\right)\right] + O(\xi^3), \tag{162}$$

so that the system is consistent of order two if and only if:

$$\alpha = \beta = 1/2 \tag{163}$$

or:

$$\theta = 1 - 1/\sqrt{2}. \tag{164}$$

If we choose $\alpha = \beta = 1/2$, we lose absolute stability. It is then convenient to choose θ according to (164).

When applying the θ-method in the context of PDEs, it is convenient that in each on the three equations appearing in (154) the equation is the same. This points to the condition:

$$\alpha\beta = \beta(1 - 2\theta), \tag{165}$$

which implies:

$$\alpha = (1 - 2\theta)/(1 - \theta); \, \beta = \theta \Big/ (1 - \theta). \tag{166}$$

Combining (166) and the condition $\alpha > \beta$, we deduce that $0 < \theta < 1/3$.

A more careful analysis shows the existence of $\theta^*(\theta^* = 0.087385580, \cdots,)$, so that for $\theta^* < \theta < 1/3$ the scheme is unconditionally and absolutely stable. With the choice $\theta = 1 - 1/\sqrt{2} = 0.292893219 \cdots$ it is also guaranteed that the scheme has order two.

6.5 Application of the splitting method to the Burgers equation

In this section we will apply the splitting θ-method described in the previous section to the viscous Burgers equation:

$$\begin{cases} u_t - \nu u_{xx} + (u^2)_x = f, & 0 < x < 1, \quad t > 0 \\ u = 0, & x = 0, \, 1, \quad\;\; t > 0 \\ u(x, \, 0) = u_0(x), & 0 < x < 1. \end{cases} \tag{167}$$

We are considering the problem in a bounded set with Dirichlet boundary conditions. Obviously, when considering the Cauchy problem in the whole real line this demands a first step in the approximation that substitutes the real line $\mathbb{R}$ by a bounded interval $[-k, k]$, with k sufficiently large. The error that is made in that approximation can be estimated using an energy method. In this way, we are reduced to the Dirichlet problem in a bounded interval, which is the problem in which we will focus in this section.

Applying the θ-method, we obtain the following sequence of equations:

$$\begin{cases} \frac{u^{k+\theta}-u^k}{\theta\Delta t} - \alpha\nu u_{xx}^{k+\theta} = f^{k+\theta} = \beta\nu u_{xx}^k - \left(\left(u^k\right)^2\right)_x, \quad 0<x<1 \\ u^{k+\theta} = 0, \; x=0,\,1, \end{cases} \tag{168}$$

$$\begin{cases} \frac{u^{k+1-\theta}-u^{k+\theta}}{(1-2\theta)\Delta t} - \beta\nu u_{xx}^{k+1-\theta} + \left(\left(u^{k+1-\theta}\right)^2\right)_x = f^{k+\theta} + \alpha\nu u_{xx}^{k+\theta}, \quad 0<x<1 \\ u^{k+1-\theta} = 0, \; x=0,\,1, \end{cases} \tag{169}$$

$$\begin{cases} \frac{u^{k+1}-u^{k+1-\theta}}{\Delta t} - \alpha\nu u_{xx}^{k+1} = f^{k+1} + \beta\nu u_{xx}^{k+1-\theta} - \left(\left(u^{k+1-\theta}\right)^2\right)_x, \quad 0<x<1 \\ u^{k+1} = 0, \; x=0,\,1. \end{cases} \tag{170}$$

We should point that (168) and (170) are reduced to solving a *linear Dirichlet problem*, that can be solved using a finite element method, and this gives a completely discrete system.

Problem (169) is non-linear, although numerical experiments indicate that essentially the same results are obtained if the non-linearity $\left(\left(u^{k+1-\theta}\right)^2\right)_x$ is substituted by $2\left(u^{k+\theta}\left(u^{k+1-\theta}\right)_x\right)$, and in that case the reduced system that is obtained has the virtue of being linear.

In the context of the Navier-Stokes equations, the θ-method has the virtue of uncoupling the non-linearity of the incompressibility condition. We thus obtain:

$$\begin{cases} \frac{u^{k+\theta}-u^k}{\theta\Delta t} - \alpha\nu\Delta u^{k+\theta} + \nabla p^{k+\theta} = f^{k+\theta} + \beta\nu\Delta u^k - \left(u^k\cdot\nabla\right)u^k \text{ en } \Omega, \\ \nabla\cdot u^{k+\theta} = 0 \text{ en } \Omega, \\ u^{k+\theta} = 0 \text{ en } \partial\Omega, \end{cases}$$

$$\begin{cases} \frac{u^{k+1-\theta}-u^{k+\theta}}{(1-2\theta)\Delta t} - \beta\nu\Delta u^{k+1-\theta} + \left(u^{k+1-\theta}\cdot\nabla\right)u^{k+1-\theta} = f^{k+\theta} + \alpha\nu\Delta u^{k+\theta} - \nabla p^{k+\theta} \text{ en } \Omega \\ u^{k+1-\theta} = 0 \text{ en } \Omega, \end{cases}$$

$$\begin{cases} \frac{u^{k+1}-u^{k+1-\theta}}{\theta\Delta t} - \alpha\nu\Delta u^{k+1} + \nabla p^{k+1} = f^{k+1} + \beta\nu\Delta u^{k+1-\theta} - \left(u^{k+1-\theta}\cdot\nabla\right)u^{k+1-\theta} \text{ in } \Omega \\ \nabla\cdot u^{k+1} = 0 \text{ in } \Omega, \\ u^{k+1} = 0 \text{ en } \partial\Omega. \end{cases}$$

7 Elliptic convection-diffusion problems

As we have seen in the previous section, when applying the splitting method to the viscous Burgers equation we obtain a family of elliptic problems with quadratic convection of the form:

$$\begin{cases} -\nu u_{xx} + (u^2)_x = f, \quad 0 < x < 1 \\ u = 0, \; x = 0, \; 1. \end{cases} \tag{171}$$

In this section we will breafly analyze the existence and uniqueness of solutions for this equation.

We will start remembering some elementary results for the linear Dirichlet problem in absence of convection:

$$\begin{cases} -\nu u_{xx} = f, \qquad 0 < x < 1 \\ u = 0, \; x = 0, \; 1. \end{cases} \tag{172}$$

In this case, it is well known that for every $f \in H^{-1}(0,1)$ there is a unique weak solution $u \in H_0^1(0,1)$ such that:

$$\begin{cases} \nu \int_0^1 u_x \varphi_x dx = \langle f, \, \varphi \rangle, \, \forall \varphi \in H_0^1(0, \, 1) \\ u \in H_0^1(0, \, 1). \end{cases} \tag{173}$$

In (173), $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(0,1)$ and $H_0^1(0,1)$. Besides, when $f \in L^2(0,1)$, the solution belongs to $H^2(0,1)$.

The weak solution can be obtaines either by a direct application of the Lax-Milgram lemma or by the direct method in the calculus of variations, minimizing the functional:

$$\begin{cases} J : \, H_0^1(0, \, 1) \longrightarrow \mathbb{R}, \\ J(v) = \frac{1}{2} \int_0^1 \mid v_x \mid^2 dx - \langle f, \, v \rangle. \end{cases} \tag{174}$$

The simplest way of addressing the non-linaer problem (171) is to use a fixed-point technique. For every $v \in H_0^1(0,1)$, we can consider the problem:

$$\begin{cases} -\nu u_{xx} = f - \left(v^2\right)_x, \, 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases} \tag{175}$$

As $f - (v^2)_x \in H^{-1}(0,1)$, problem (**??**) admits a unique solution $u \in H_0^1(0,1)$. This allows us to define a non-linear map $\mathcal{N} : H_0^1(0,1) \to H_0^1(0,1)$ that to every $v \in H_0^1(0,1)$ associates $\mathcal{N}v = u$. It is not hard to check that $\mathcal{N}$ is a compact mapping. In order to see this, it is enough to check that if v is in a compact subset of $H_0^1(0,1)$, then $(v^2)_x = 2vv_x$ is in a compact subset of $L^2(0,1)$ ans thus in a compact subset of $H^{-1}(0,1)$. It is thus natural to apply Schauder's theorem. But this is not immediately possible. Indeed:

$$\begin{aligned} \nu \parallel \mathcal{N}(v) \parallel^2_{H_0^1(0,1)} &= \parallel f - \left(v^2\right)_x \parallel_{H^{-1}(0,1)} \leq \parallel f \parallel_{H^{-1}(0,1)} + \parallel v^2 \parallel_{L^2(0,1)} \\ &\leq \parallel f \parallel_{H^{-1}(0,1)} + C \parallel v \parallel^2_{H_0^1(0,1)} . \end{aligned} \tag{176}$$

Thus:

$$\| u \|_{H_0^1(0,1)} \leq \frac{\| f \|_{H^{-1}(0,1)}}{\nu} + \frac{C}{\nu} \| v \|_{H_0^1(0,1)}^2 . \tag{177}$$

With the aim of applying Schauder's theorem, the simplest approach is to find a ball B_R of $H_0^1(0,1)$ in which the map $\mathcal{N}$ is invariant. In virtue of the estimate (177), this demands:

$$\frac{\| f \|_{H^{-1}(0,1)}}{\nu} + \frac{C}{\nu} R^2 \leq R, \tag{178}$$

which is possible for any $f \in H^{-1}(0,1)$ either if $\nu > 0$ is sufficiently large or if $\nu > 0$ is arbitrary, as long as $\|f\|_{H^{-1}(0,1)}$ is sufficiently small compared to ν.

This point of view, however, does not allow to solve the nonlinear problem (171) in its whole generality. In order to do so, we introduce a bounded approximation of the quadratic nonlinearity appearing in (171), i.e.:

$$\phi_k(s) = \min(s^2, k). \tag{179}$$

We will now consider, instead of (171), the problem with a truncated nonlinearity:

$$\begin{cases} -\nu u_{xx} + (\phi_k(u))_x = f, \quad 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases} \tag{180}$$

In this case, the previous argument allows to conclude the existence of a solution $u_k \in H_0^1(0,1)$. Indeed, in the present case, the condition over the radius R of the ball B_R that is demanded so that Schauder's fixed point theorem can be applied is simply:

$$\frac{\| f \|_{H^{-1}(0,1)}}{\nu} + \frac{Ck^2}{\nu} \leq R, \tag{181}$$

which is obviously true if $R > 0$ is sufficiently large.

On the other hand, it is not hard to obtain a uniform bound over the sequence of approximate solutions $\{u_k\}_{k \geq 0}$. Indeed, multiplying in (180) by u_k and integrating by parts or, in other words, using the function u_k as a test function in the weak formulation of (180), we obtain:

$$\nu \int_0^1 |u_{k,x}|^2 \, dx - \int_0^1 \phi_k(u_k) u_{k,x} dx = \langle f, u_k \rangle. \tag{182}$$

Now, as:

$$\phi_k(u_k) u_{k,x} = \frac{\partial}{\partial x} \left(\psi_k \left(u_k \right) \right)$$

where:

$$\psi_k(s) = \int_\sigma^s \phi_k(\sigma) d\sigma,$$

we have:

$$\int_0^1 \phi_k(u_k)u_{k,\,x}dx = \int_0^1 \frac{\partial}{\partial x}\left(\psi_k\left(u_k\right)\right)dx = 0. \tag{183}$$

and thus:

$$\nu \int_0^1 |u_{k,\,x}|^2\, dx = \langle f,\, u_k\rangle \tag{184}$$

which implies the bound:

$$\| u_k \|_{H_0^1(0,\,1)} \leq \frac{1}{\nu} \| f \|_{H^{-1}(0,\,1)},\ \forall k \geq 1. \tag{185}$$

This allows to take the limit in the sequence of approximate solutions. Indeed, as $\{u_k\}$ is a bounded sequence in $H_0^1(0,1)$, we can obtain a subsequence, which we will also denote by $\{u_k\}$, such that:

$$u_k \rightharpoonup u \text{ weakly in } H_0^1(0,\,1). \tag{186}$$

Furthermore, we can extract this subsequence so that:

$$u_k \longrightarrow u \text{ strongly in } L^2(0,\,1) \tag{187}$$

and:

$$u_k \longrightarrow u \quad x \in (0,\,1). \tag{188}$$

This allows to take the limit in the weak formulation of (180):

$$\int_0^1 u_{k,\,x}\varphi_x dx - \int_0^1 \phi_k(u_k)\varphi_x dx = \langle f,\, \varphi\rangle,\ \forall \varphi \in H_0^1(0,\,1). \tag{189}$$

Indeed, taking the limit in (189) we obtain that the limit $u \in H_0^1(0,1)$ is a weak solution of (171), since it satisfies:

$$\int_0^1 u_x\varphi_x dx - \int_0^1 u^2\varphi_x dx = \langle f,\, \varphi\rangle,\ \forall \varphi \in H_0^1(0,\,1). \tag{190}$$

In virtue of the weak convergence in $H_0^1(0,1)$ of $\{u_k\}$, the unique difficulty to obtain (190) from (189) is the pass to the limit in the non-linear term. This can be done by checking that:

$$\phi_k(u_k) \rightharpoonup u^2 \text{ weakly in } L^2(0,\,1).$$

This holds because, as a consequence of (188):

$$\phi_k(u_k) \longrightarrow u^2,\ x \in (0,\,1)$$

and, on the other hand:

$$||\phi_k(u_k)||_{L^\infty(0,\,1)} \leq \left|\left|u_k^2\right|\right|_{L^\infty(0,\,1)} \leq \| u_k \|^2_{L^\infty(0,\,1)} \leq C||u \|^2_{H_0^1(0,\,1)} \leq C. \tag{191}$$

At this point, we have used the following classical convergence result that is proven using Egorov's theorem:

Lemma 7.1. *Let Ω be a bounded domain of $\mathbb{R}^n$. Let $h_k : \Omega \to \mathbb{R}$ be a sequence of measurable functions such that:*

$$\| h_k \|_{L^p(\Omega)} \leq C, \, \forall k \geq 1$$

with $p > 1$ and:

$$h_k \longrightarrow h, \, x \in \Omega.$$

Then:

$$h_k \longrightarrow h \text{ in } L^q(\Omega), \, \forall 1 \leq q < p$$

and:

$$h_k \longrightarrow h \text{ weakly in } L^p(\Omega),$$

if $p < \infty$ or weakly-$*$ in L^∞ if $p = \infty$.

Thanks to the previous argument combining the approximation by truncated functions and the pass to the limit, we have proven the existence of at least one solution of (171) for each $f \in H^{-1}(0,1)$ and every $\nu > 0$.

Let's now see that this solution is unique.

As it's usual in these cases, we will suppose that there exist two functions u_1, u_2, we will define $v = u_1 - u_2$ and we will try to prove that $v \equiv 0$. The function v satisfies:

$$\begin{cases} -\nu v_{xx} + \left(u_1^2 - u_2^2\right)_x = 0, \quad 0 < x < 1, \\ \quad v(0) = v(1) = 0, \end{cases} \tag{192}$$

Although the uniqueness result is true for every $\nu > 0$, in this notes we will prove it only for values of $\nu > 0$ that are sufficiently large.

Multiplyin in (192) by v and integrating by parts we obtain:

$$\begin{aligned} \nu \int_0^1 v_x^2 dx &= \int_0^1 (u_1 + u_x) \, v v_x dx \\ &\leq \| u_1 + u_2 \|_{L^\infty(0,1)} \| v \|_{L^2(0,1)} \| v_x \|_{L^2(0,1)} \end{aligned}$$

from which we deduce that:

$$\nu \| v \|_{H_0^1(0,1)} \leq C \| u_1 + u_2 \|_{L^\infty(0,1)} \| v \|_{H_0^1(0,1)}$$

where $C > 0$ is a constant independent of ν. In other words:

$$\nu \leq C \| u_1 + u_2 \|_{L^\infty(0,1)} . \tag{193}$$

Now, any solution of (171) satisfies:

$$\nu \int_0^1 u_x^2 dx = \langle f, \, u \rangle.$$

This is the case since:

$$\int_0^1 u^2 u_x dx = 0.$$

And thus:

$$\nu \parallel u_x \parallel_{H_0^1(0,1)} \leq \parallel f \parallel_{H^{-1}(0,1)},$$

so we have:

$$\nu \parallel u \parallel_{L^\infty(0,1)} \leq C \parallel f \parallel_{H^{-1}(0,1)} . \tag{194}$$

Combining (193) and (194) we deduce that:

$$\nu \leq \frac{C}{\nu} \parallel f \parallel_{H^{-1}(0,1)},$$

which is impossible if $\nu > 0$ is sufficiently large.

This proves uniqueness if $\nu > 0$ is sufficiently large.

The same arguments allow to prove existence and uniqueness (for values of $\nu > 0$ sufficiently large) in the case of the solutions of the stationary Navier-Stokes problem:

$$\begin{cases} -v\Delta u + (u \cdot \nabla)u + \nabla p = f & \text{in} \quad \Omega \\ \nabla \cdot u = 0 & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \partial\Omega. \end{cases}$$

This justifies the existence and uniqueness of approximate solutions obtained using the splitting θ-method in the previous section.

8 Non-linear semigroups

We have applied "splitting" or decomposition methods for discrete-time equations, inspired from the classical numerical approximation theory of ordinary differential equations.

There are two main reasons for considering discrete time schemes. On the one hand, the actual numerical solution of a differential problems always implies a discretization in time. It is then natural to consider discrete time schemes. On the other hand, there are many ocasions in which the use of time discretizations is also a method to obtain a solution for the continuous time model.

In the context of linear PDE, there are many techniques of obtaining solutions available: fundamental solutions, Fourier analysis, separation of variables, spectral methods, semigroups... However, non-linear equations are much more complex and systematic methods of finding solutions are less available. In this section, we will illustrate the possible use of time discretization methods in the model case of a parabolic non-linear equation involving the p-Laplacian operator:

$$\begin{cases} u_t - \left(\mid \nabla u \mid^{p-2} \nabla u \right) = 0 & \text{in} \quad \Omega \times (0, \infty) \\ u = 0 & \text{at} \quad \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in} \quad \Omega. \end{cases} \tag{195}$$

Throughout this section, we will suppose that Ω is a regular bounded open subset of $\mathbb{R}^n$ and that $p \geq 2$, although the model under consideration makes sense also for $p \geq 1$, $p = 1$ being a limit case. However, in order to avoid additional technical difficulties, we will assume that $p \geq 2$.

It's convenient to note that, when $p = 2$, (195) is reduced to the linear heat equation for which, as we said above, there are several different methods available. The non-linear case, as we will see, is considerably more complex.

From the numerical point of view, the most natural way to address this problem is either via a Galerkin method or using a time discretization. In this section we will analyze both methods, but we will first briefly analyze the underlying elliptic problem.

8.1 The elliptic problem

Given $f = f(x)$, we consider the elliptic problem:

$$\begin{cases} -\left(\mid \nabla u \mid^{p-2} \nabla u\right) = f & \text{in} \quad \Omega \\ u = 0 & \text{at} \quad partial\Omega. \end{cases} \tag{196}$$

Its variational formulation is given by:

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \int_\Omega \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx, \forall \varphi \in W_0^{1,p}(\Omega). \end{cases} \tag{197}$$

In order for this variational formulation to make sense, it is enough for f to belong to the dual space of $W_0^{1,p}(\Omega)$. It suffices then to have $f \in L^q(\Omega)$, with $q > np/(np + p - 1)$.

In order to obtain a weak solution, we consider the functional:

$$J : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}, \tag{198}$$

such that:

$$J(v) = \frac{1}{p} \int_\Omega \mid \nabla v \mid^p dx - \int_\Omega f v dx. \tag{199}$$

The space $W_0^{1,p}(\Omega)$ is reflexive. On the other hand, J is continuous, convex and coercive. Thus, the functional J attains its minimum in a point $u \in W_0^{1,p}(\Omega)$. It is easy to check that u is a solution of (196) in the sense of (197).

On the other hand, the weak solution is unique, since J is strictly convex.

Another way of checking uniqueness is, as usual, to suppose that there are two solutions u_1 and u_2 and to consider $v = u_! - u_2$. Then, v satisfies:

$$\begin{cases} -\left(\mid \nabla u_1 \mid^{p-2} \nabla u_1 - \mid \nabla u_2 \mid^{p-2} \nabla u_2\right) = 0 & \text{in} \quad \Omega \\ u_1 - u_2 = 0 & \text{at} \quad \partial\Omega, \end{cases} \tag{200}$$

or, in other words, its variational version:

$$\int_\Omega \left[\mid \nabla u_1 \mid^{p-2} \nabla u_1 - \mid \nabla u_2 \mid^{p-2} \nabla u_2\right] \cdot \nabla \varphi = 0 \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{201}$$

Taking $\varphi = u_1 - u_2 = v$ as a test function, we obtain:

$$\int_\Omega \left[\mid \nabla u_1 \mid^{p-2} \nabla u_1 - \mid \nabla u_2 \mid^{p-2} \nabla u_2 \right] \cdot \nabla \left(u_1 - u_2 \right) dx = 0. \tag{202}$$

On the other hand, we have the following inequality in $\mathbb{R}^n$:

$$\left(\mid a \mid^{p-2} a - \mid b \mid^{p-2} b \right) \cdot (a - b) \geq 0, \, \forall a, \, b \in \mathbb{R}^n, \tag{203}$$

from which we deduce that, necessarily:

$$\left(\mid \nabla u_1 \mid^{p-2} \nabla u_1 - \mid \nabla u_2 \mid^{p-2} \nabla u_2 \right) \cdot \left(\nabla u_1 - \nabla u_2 \right) = 0 \, x \in \Omega. \tag{204}$$

But we can say actually more and guarantee that: Pero en realidad podemos decir más aún y garantizar que

$$\left(\mid a \mid^{p-2} a - \mid b \mid^{p-2} b \right) \cdot (a - b) > 0 \text{ if } a \neq b \tag{205}$$

which ensures, in cirtue of (202), that:

$$\nabla u_1 = \nabla u_2 \, x \in \Omega. \tag{206}$$

Noting that $u_1, u_2 \in W_0^{1,p}(\Omega)$, we then deduce that $u_1 = u_2$ in Ω.

8.2 The Galerkin method

The Galerkin method to approximate equation (195) is introduced in the same way as in the linear case. We introduce an approximation of $W_0^{1,p}(\Omega)$ by finite dimensional subspaces V_h of finite elements P_1 piecewise and continuous. The reader who is interested in the basic aspects of the finite element method can check [35].

We assume that:

$$\dim(V_h) = N_h, \, V_h = (e_1, \cdots, e_{N_h}). \tag{207}$$

We will then look for approximations of the solution of (195) such that:

$$u_h \in C\left([0, \, \infty); \, V_h \right) \tag{208}$$

so that:

$$u_h(x, \, t) = \sum_{j=1}^{N_h} u_j(t) e_j(x). \tag{209}$$

In order to introduce this approximation, it is convenient to previously introduce the variational formulation of (195):

$$\begin{cases} u \in C\left([0, \, \infty); \, L^2(\Omega) \right) \cap L^p \left(0, \, \infty; \, W_0^{1,p}(\Omega) \right) \\ \frac{d}{dt} \int_\Omega u(x, \, t) \varphi(x) dx + \int_\Omega \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla \varphi dx = 0, \, \forall \varphi \in W_0^{1,p}(\Omega), \\ \int_\Omega u(x, \, t) \varphi(x) dx \to \int_\Omega u_0(x) \varphi(x) dx, \, t \to 0, \, \forall \varphi \in W_0^{1,p}(\Omega). \end{cases} \tag{210}$$

The space in which we are searching for a solution is inspired in the energy estimate for the solutions of (195) which, formally, consists on multiplying equation (195) by u and integrating by parts in Ω. We thus obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,\,t)dx = -\int_\Omega \mid \nabla u \mid^p dx. \tag{211}$$

Integrating in time we obtain:

$$\int_\Omega u^2(x,\,t)dx + \int_0^t \int_\Omega \mid \nabla u \mid^p dxdt = \int_\Omega u_0^2(x)dx. \tag{212}$$

From this formal estimate we deduce that, if the initial datum $u_0 \in L^2(\Omega)$, then it is expected that the solution will verify $u \in L^\infty(0,\infty; L^2(\Omega)) \cap L^p(0,\infty; W_0^{1,p}(\Omega))$.

From (210), it is easy to introduce a Galerkin approximation:

$$\begin{cases} u_h \in C([0,\,\infty);\, V_h) \\ \frac{d}{dt}\int_\Omega u_h\varphi dx + \int_\Omega \mid \nabla u_h \mid^{p-2} \nabla u_h \cdot \nabla\varphi dx = 0,\, \forall\varphi \in V_h \\ u_h(0) = u_{0,\,h}, \end{cases} \tag{213}$$

where $u_{0,h}$ is an approximation of u_0 in V_h satisfying:

$$u_{0,\,h} \to u_0 \text{ in } L^2(\Omega). \tag{214}$$

As usual in the context of Galerkin approximations, we have to address two questions. First, we have to prove that (213) admits a unique solution u_h, and then we have to prove that it converges to a solution of (195).

With the aim of verifying if (213) admits a solution, we will write this variational formulation as a system of N_h non-linear ordinary differential equations with N_h unknowns. In virtue of the structure of (209) and noting that (213) is satisfied for every test function $\varphi \in V_h$ if and only if it is satisfies for every function e_j, $j = 1, \ldots, N_h$ in the basis of V_h, we deduce that (213) is equivalent to:

$$\begin{cases} MU' + F(U) = 0, \quad t > 0 \\ U(0) = U_{0,\,h}, \end{cases} \tag{215}$$

where the column vector $U = (u_1, \ldots, u_{N_h})'$ codifies the N_h unknowns of the system, and $U_{o,h}$ represents in the same way the initial datum $u_{0,h} \in V_h$, M is the mass matrix of the finite elements method, $M = (m_{ij})_{1\le i,j\le N_h}$ with:

$$m_{ij} = \int_\Omega e_i(x)e_j(x)dx, \tag{216}$$

and F is a non-linear function defined in the following way:

$$F(U) = (F_j(U))_{1\le j\le N_h}, \tag{217}$$

where:

$$F_j(U) = \int_\Omega \left| \nabla \left(\sum_{k+1}^{N_h} u_{j_k} e_k(x) \right) \right|^{p-2} \nabla \left(\sum_{k+1}^{N_h} u_k e_k(x) \right) \cdot \nabla e_j(x) dx. \tag{218}$$

In the case of $p = 2$, $F(U) = RU$, where R is the stiffness matrix of the finite element method.

When $p \geq 2$, the function F is non-linear and of class C^2, so that (215) admits a unique local solution. With the aim of proving that this solution is global, we need an a priori estimate. Applying again the energy method that consists in taking in (213) the solution u_h as a test function or taking the scalar product in (215) by the unknown U, the obtain that (212) is also satisfies for each approximation u_h. We thus deduce that the solution of (213) is globally defined.

This also yields a uniform bound on the approximate solutions:

$$\frac{1}{2} \parallel u_h \parallel^2_{L^\infty(0,\,\infty;\,L^2(\Omega))} + \parallel \nabla u_h \parallel^p_{L^p(\Omega\times(0,\,\infty))} \leq \frac{1}{2} \parallel u_{0,\,h} \parallel^2_{L^2(\Omega)},\ \forall h > 0. \tag{219}$$

From this estimate we can deduce that, extracting subsequences:

$$\begin{cases} u_h \rightharpoonup u & \text{weakly -}* \text{ in} \quad L^\infty\left(0,\,\infty;\,L^2(\Omega)\right) \\ u_h \rightharpoonup u & \text{weakly in} \quad L^p\left(0,\,\infty;\,W_0^{1,\,p}(\Omega)\right). \end{cases} \tag{220}$$

However, these weak convergences are not sufficient to pass to the limit in (213), because it is a non-linear problem.

The pass to the limit needs additional estimates. In order to obtain them, it is convenient to go back for a moment to the continuous equation and to check any other estimate that hold. First we observe that, if u_1 and u_2 are solutions of (195), then multiplying by $u_1 - u_2$ the equation that $u_1 - u_2$ satisfies we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1 - u_2|^2\,dx + \int_\Omega \left(|\nabla u_1|^{p-2}\,\nabla u_1 - |\nabla u_2|^{p-2}\,\nabla u_2 \right) \cdot (\nabla u_1 - \nabla u_2)\,dx = 0. \tag{221}$$

We now use the existence of a $c > 0$ such that:‖

$$\left(\mid a \mid^{p-2} a - \mid b \mid^{p-2} b\right) \cdot (a - b) \geq c \mid a - b \mid^p,\ \forall a,\, b \in \mathbb{R}^n. \tag{222}$$

We thus obtain that:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_1 - u_2|^2\,dx + c\int_\Omega |\nabla\,(u_1 - u_2)|^p\,dx \leq 0. \tag{223}$$

And we deduce:

$$\frac{1}{2}\int_\Omega |u_1(x,\,t) - u_2(x,\,t)|^2\,dx + c\int_{\Omega\times(0,\,t)} |\nabla u_1 - \nabla u_2|^p\,dxds \leq \frac{1}{2}\int_\Omega |u_{1,\,0} - u_{2,\,0}|^2\,dx. \tag{224}$$

‖The proof of this estimate is left as an exercise to the reader.

From this inequality we deduce that, if (u_j) is a sequence of solutions of (195) such that their initial data $(u_{j,0})$ are a Cauchy sequence in $L^2(\Omega)$, then (u_j) is also a Cauchy sequence in $L^\infty(0,\infty;L^2(\Omega)) \cap L^p(0,\infty;W_0^{1,‘}(\Omega))$.

This argument, applied in the context of Galerkin approximations, allows to prove that ∇u_h is a Cauchy sequence in $L^p(\Omega \times (0,\infty))$ which, in virtue of (220), yields the strong convergence:

$$\nabla u_h \longrightarrow \nabla u \text{ en } L^p(\Omega \times (0,\,\infty)), \tag{225}$$

which is needed to pass to the limit in the Galerkin approximation (213) and obtain in the limit the variational formulation (210) of equation (195).

This is rigorously true when the Galerkin spaces V_h grow as h decreases. This is not the case for an arbitrary triangulation in the context of finite elements, but it is true if, as h decreases, the new space V_h is obtained by refininf the mesh of a previous space.

We have thus proven that the Galerkin method yields a way of constructing solutions for (195).

8.3 Time discretization

Given a time step $\Delta t > 0$, which will tend to zero, we want to obtain solutions to (195) as the limit as $\Delta t \to 0$ of solutions of discrete-time systems.

In the following, with the aim of simplifying the notation, we will denote by A_p the p-Laplacian operator, so that:

$$A_p(u) = -\left(\mid \nabla u \mid^{p-2} \nabla u\right). \tag{226}$$

When discretizing (195), we have two basic choices:

- The explicit Euler scheme:

$$\begin{cases} u^{k+1} + \Delta t A_p(u^k) = u^k, & \text{in} \quad \Omega, \\ u^{k+1} = 0 & \text{at} \quad \partial\Omega,\ k \geq 0, \\ u^0 = u_0 & \text{in} \quad \Omega. \end{cases} \tag{227}$$

- The implicit Euler scheme:

$$\begin{cases} u^{k+1} + \Delta t A_p\left(u^{k+1}\right) = u^k, & \text{in} \quad \Omega, \\ u^{k+1} = 0 & \text{at} \quad \partial\Omega,\ k \geq 0 \\ u^0 = u_0, & \text{in} \quad \Omega. \end{cases} \tag{228}$$

In both cases, u^k represents an approximation to the solution of (195) at the instant fo time $t = k\Delta t$. Also, in both schemes the initial datum u_0 of system (195) is taken as the initial value of the iteration for $k = 0$.

The main advantage of the explicit scheme (227) over the implicit scheme (228) is that the succesive values of u^k are obtained without need of solving any equations, just reading their values from the values of the previous step. Thus, from (227) we have:

$$u^{k+1} = u^k - \Delta t A_p\left(u^k\right) = B_{\Delta_t}\left(u^k\right). \tag{229}$$

Iterating this non-linear mapping $B_{\Delta t}$, we can write the value of the discrete solution from the initial datum in an explicit form:

$$u^k = \left(B_{\Delta_t}\right)^k (u_0)\,. \tag{230}$$

It is however convenient to point out that the expression (230), besides being strongly non-linear, involves the derivatives of the initial data u_0 up to order $2k$. Noting that the number of steps k necessary to approximate the value of the solution at a fixed instant of time T tends to infinity as $\Delta t \to 0$, this method can only be useful when the initial data u_0 is of the class C^∞. However, even in this case the convergence of the method is not guaranteed.

In this context, it is much more natural to use the implicit method that, as we know, in the context of ordinary differential equaitons is unconditionally stable and convergent. Now, the application of the implicit scheme (228) demands, in each step, to solve the elliptic non-linear problem:

$$\begin{cases} u^{k+1} + \Delta t A_p\left(u^{k+1}\right) = u^k, & \text{in} \quad \Omega, \\ u^{k+1} = 0 & \text{at} \quad \partial\Omega. \end{cases} \tag{231}$$

The solution to this problem can be obtained using techniques that are analogous to the ones obtained in section (8.1). More concretely, in order to prove the existence of a solution to (231) it is enough to minimize the functional:

$$J_k(v) = \frac{\Delta t}{p}\int_\Omega \mid \nabla v \mid^p dx + \frac{1}{2}\int_\Omega v^2 dx - \int_\Omega u^k v dx, \tag{232}$$

in the space $W_0^{1,p}(\Omega)$. The uniqueness of the solution is then proven using analogous techniques to the ones of section (8.1).

Besides, the energy method yields immediate estimates. Indeed, multiplyinh in (231) by u^{k+1} and integrating in Ω we obtain:

$$\begin{array}{rcl} \int_\Omega \left|u^{k+1}\right|^2 dx + \Delta t \int_\Omega \left|\nabla u^{k+1}\right|^p dx & = & \int_\Omega u^{k+1} u^k dx \\ & \le & \left(\int_\Omega \left|u^{k+1}\right|^2\right)^{1/2} \left(\int_\Omega \left|u^k\right|^2 dx\right)^{1/2}, \end{array} \tag{233}$$

from which we deduce, in particular:

$$\left|\left|u^{k+1}\right|\right|_{L^2(\Omega)} \le \left|\left|u^k\right|\right|_{L^2(\Omega)}. \tag{234}$$

Iterating this inequality we easily obtain the bound:

$$\max_k \left|\left|u^k\right|\right|_{L^2(\Omega)} \le ||u_0||_{L^2(\Omega)}\,, \tag{235}$$

independent of Δt.

However, from (233) we also deduce that:

$$\begin{aligned} \Delta t \sum_k \int_\Omega \left|\nabla u^{k+1}\right|^p dx &\leq \sum_k \left|\left|u^{k+1}\right|\right|_{L^2(\Omega)} \left(\left|\left|u^k\right|\right|_{L^2(\Omega)} - \left|\left|u^{k+1}\right|\right|_{L^2(\Omega)}\right) \\ &\leq \left|\left|u_0\right|\right|_{L^2(\Omega)} \sum_k \left(\left|\left|u^k\right|\right| - \left|\left|u^{k+1}\right|\right|_{L^2(\Omega)}\right) \\ &\leq \left|\left|u_0\right|\right|^2_{L^2(\Omega)} . \end{aligned} \tag{236}$$

The estimates (235) and (236) are obviously the discrete analogues of the energy estimate.

From the discrete solution $\{u^k\}_{k\geq 0}$ we can construct a continuous function $u_{\Delta t}(x,t)$ that in each time interval of the form $[k\Delta t, (k+1)\Delta t]$ takes the value of $u^k(x)$. We thus obtain a family $\{u_{\Delta t}\}_{\Delta t>0}$ of continuous functions, bounded in the space $L^\infty(0,\infty; L^2(\Omega)) \cap L^p(0,\infty; W_0^{1,p}(\Omega))$. Extracting subsequences we can take the limit weakly in the sequence $u_{\Delta t}$ as $\Delta t \to 0$ and thus obtain a limit function $u(x,t)$. The problem that persists is to prove that this limit is a solution to the non-linear equatino (195) or, in other words, that it verifies the weak formulation (210). Again, due to the non-linear character of the problems under consideration, this is not completely immediate.

With the aim of guaranteeing that the limit u is a solution to (195) or (210) we need to prove the strong convergence of $\nabla u_{\Delta t}$ in $L^1_{\text{loc}}(0,\infty; L^{p-1}(\Omega))$. The key ingredient to do so is to obtain higher order estimates. We will again go back to the continuous problem to understand how these estimates can be obtained. Multiplying in (195) by u_t and integrating in Ω we obtain:

$$\int_\Omega u_t^2 dx + \int_\Omega \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla u_t dx = 0.$$

On the other hand:

$$\int_\Omega \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla u_t dx = \frac{1}{p}\frac{d}{dt}\int_\Omega \mid \nabla u \mid^2 dx.$$

We thus obtain the identity:

$$\frac{d}{dt}\left[\frac{1}{p}\int_\Omega \mid \nabla u \mid^p dx\right] = -\int_\Omega u_t^2 dx = -\int_\Omega \left|\left(\mid \nabla u \mid^{p-2} \nabla u\right)\right|^2 dx \tag{237}$$

which, integrated in t, yields:

$$\frac{1}{p}\int_\Omega \mid \nabla u(x,\, t) \mid^p dx + \int_0^t \int_\Omega \left|\left(\mid \nabla u \mid^{p-2} \nabla u\right)\right|^2 dx ds = \frac{1}{p}\int_\Omega |\nabla u_0(x)|^p dx, \tag{238}$$

as long as $u_0 \in W_0^{1,p}(\Omega)$. We thus deduce an estimate for the continuous solution in $L^\infty(0,\infty; W_0^{1,p}(\Omega))$, in addition to a bound of $|\nabla u|^{p-2}\nabla u$ in $L^2(\Omega \times (0,\infty))$.

This estimate can be reproduced in the discrete context. Multiplying in (231) by $(u^{k+1}-u^k)/\Delta t$ we deduce that:

$$\int_\Omega \left| \frac{u^{k+1}-u^k}{\Delta t} \right|^2 dx + \frac{1}{\Delta t} \int_\Omega \left(\mid \nabla u^{k+1} \mid^{p-2} \nabla u^{k+1} \right) \cdot \left(u^{k+1}-u^k \right) dx = 0,$$

which, after the integration by parts, yields:

$$\int_\Omega \left| \frac{u^{k+1}-u^k}{\Delta t} \right|^2 dx + \frac{1}{\Delta t} \int_\Omega \mid \nabla u^{k+1} \mid^p dx - \frac{1}{\Delta t} \int_\Omega \mid \nabla u^{k+1} \mid^{p-2} \nabla u^{k+1} \cdot \nabla u^k dx = 0. \quad (239)$$

From this expression it is easy to deduce that:

$$\int_\Omega \mid \nabla u^{k+1} \mid^p dx \leq \int_\Omega \mid \nabla u^k \mid^p dx, \quad (240)$$

and, iterating this inequality, yields the bound:

$$\| \nabla u^k \|_{L^p(\Omega)} \leq \| \nabla u_0 \|_{L^p(\Omega)}, \; \forall k \geq 0, \quad (241)$$

which is independent of Δt.

Once we have this bound it is easy to conclude that:

$$\Delta t \sum_k \int_\Omega \left| \frac{u^{k+1}-u^k}{\Delta t} \right|^2 dx \leq C \| \nabla u_0 \|^p_{L^p(\Omega)} . \quad (242)$$

We thus obtain a discrete analogue of the second order energy estimate (238).

This estimate allows to obtain the strong convergence of $\nabla u_{\Delta t}$ which is needed to prove that the limit u is a olution to equation (195), although in this section we will omit the details.

8.4 Conclusion

In this section we have seen how the implicit Euler time discretization and the continuous time Galerkin approximation are two approximation methods for non-linear parabolic equations of the form (195).

The fact that the discrete time scheme is a natural way of approximating the evolution PDE is what motivated us in the previous section to introduce the “splitting” or decomposition methods of discrete time equations.

The developments of this section, in the context of the p-Laplacian heat equation (195), can also be done, with analogous arguments, for other models as the two-dimensional Navier-Stokes equations or the one-dimensional viscous Burgers equation.

As the system (195) si autonomous and admits unique solutions, it is the generator of a semigroup that, in analogy to the linear case, is denoted b $S(t)$. Thus, the solution

u of (195) at the instant of time t can be denoted as $u(t) = S(t)u_0$, where $S(t)$ is the non-linear solution map. The family of maps $\{S(t)\}_{t>0}$ satisfies the two relations:

$$\begin{cases} S(0) = Id, \\ S(t) \circ S(s) = S(t+s), \end{cases}$$

and thus it is said to be a semigroup.

What we have done in this section is in the basis of the theory of non-linear semiogroups, in the context of which (195) and the other models that we have mentioned are just concrete examples that can be addressed using this systematic techinique.

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