UIMP, Santander, July 04

The system of thermoelasticity

Enrique Zuazua
Universidad Autónoma
28049 Madrid, Spain
enrique.zuazua@uam.es

E. Z., JMPA, 1995; G. LEBEAU & E. Z., ARMA, 1999; X. ZHANG, &E. Z., CCM, 2003; N. BURQ & G. LEBEAU, AENSP, 2002.

The system of thermoelasticity is an improved version of the Lamé's system for elastic bodies, in which one also takes into account thermal effects. This system couples a hyperbolic equation and a parabolic one in the same domain:

Given a bounded domain $\Omega \subset \mathbb{R}^n$ $\partial\Omega$, the system of thermoelasticity reads as follows:

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u + \alpha\nabla\theta = 0 & \text{in} \quad \Omega \times (0, \infty) \\ \theta_t - \Delta\theta + \beta \operatorname{div} u_t = 0 & \text{in} \quad \Omega \times (0, \infty) \\ u = 0, \theta = 0 & \text{on} \quad \partial\Omega \times (0, \infty) \\ u(x, 0) = u^0(x), u_t(x, 0) = u^1(x), \theta(x, 0) = \theta^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

u is the displacement vector and θ denotes the temperature.

The equations indicate that:

* The gradient of the temperature acts as a force on the Lamé system governing the elastic component;

* Pressure waves act as a heat source on the heat equation.

The finite energy state space for (1) reads

$$H = (H_0^1(\Omega))^n \times (L^2(\Omega))^n.$$

The energy, which is the addition of the elastic and the thermal energies,

$$E(t) = \frac{1}{2}\int_\Omega [\mu \mid \nabla u \mid^2 + (\lambda+\mu) \mid \mathrm{div}\, u \mid^2 + \mid u_t \mid^2 + \frac{\alpha}{\beta} \mid \theta \mid^2] dx \qquad (2)$$

decreases along trajectories. More precisely,

$$\frac{dE(t)}{dt} = -\frac{\alpha}{\beta}\int_\Omega \mid \nabla\theta(x,t) \mid^2 dx \le 0. \qquad (3)$$

C. Dafermos (1968) studied the problem of whether the energy of every solutions converges to zero as $t \to \infty$, i.e.

$$E(t) \to 0, \quad \text{as} \quad t \to \infty. \qquad (4)$$

He proved that this holds if and only if the domain Ω satisfies the following spectral condition:

$$(C) \qquad \begin{array}{l} \text{If } \varphi \in (H_0^1(\Omega))^n \text{ is such that} \\ \qquad \begin{cases} -\Delta\varphi = \gamma^2\varphi & \text{in} \quad \Omega \\ \operatorname{div}\varphi = 0 & \text{in} \quad \Omega \\ \varphi = 0 & \text{on} \quad \partial\Omega \end{cases} . \\ \text{for some } \gamma \in \mathbf{R}, \text{ then } \varphi \equiv 0. \end{array} \qquad (5)$$

Condition (C) guarantees that the Lamé system has no eigenfunction with null divergence. This condition holds generically with respect to the domain Ω . It holds for instance when the spectrum of the Dirichlet Laplacian in Ω is simple. But it fails when the domain Ω is ball or a disk. The problem of fully characterizing the domains for which it holds is open (Pompeiu's problem).

EXERCISES:

* Show that, if the Dirichlet spectrum is simple, there is no eigenvector field with null divergence.

* Apply LaSalle's invariance principle and show that, if there are no eigenvector fields with null divergence then all solutions tend to zero.

Hint: The only non dissipated solution is that for which $\nabla\theta \equiv 0$. But then, due to the Dirichlet b. c., $\theta \equiv 0$. Then, $\mathrm{div}\, u_t \equiv 0$. Then there is a solution of the Lamé system of null divergence. Using the Fourier expansion of solutions and the orthogonality of complex exponentials in infinite time this implies that this kind of solution may only exist if there is an eigenvector field with null divergence.

* Learn the main ingredients of the proof of the generic simplicity of the Dirichlet laplacian: Baire's lemma and differentiation of eigenvalues with respect to the shape of a domain, as Hadamard did.

Let us now focus on the case of the domain Ω for which (C) holds. Most of the domains do it.

In those cases $E(t) \to 0$ as $t \to \infty$ for each solution. But, is the decay rate uniform?

$$E(t) \leq Ce^{-\omega t} E(0), \; \forall t > 0 \tag{6}$$

This holds if and only if

$$E(0) \leq C \int_0^T \int_\Omega | \nabla \theta |^2 \, dxdt. \tag{7}$$

This kind of inequality arises frequently in control and inverse problems: The question is whether partial measurements of solutions allow getting uniform bounds on the whole solution.

As we have seen in the context of multi-structures, the way the differnt components of a wave-like system may interact may be rather complex. Thus, determining whether measuremts in some of the components do provide global information on solutions or not is not an easy problem.

In finite space dimensions this was done by Kalman:

$$\begin{cases} \dot{x} = Ax, & t > 0, \\ x(0) = x^0, \end{cases} \tag{8}$$

with state $x = (x_1(t), \ldots, x_N(t))^t$. The matrix A has constant coefficients and dimension $N \times N$.

Consider now a matrix B of dimensions $N \times M$.

Then,

$$\int_0^T |x(t)|^2 dt \leq C \int_0^T |Bx(t)|^2 dt,$$

iff

$$\text{rank } \left[B \,|\, BA \,|\, \cdots \,|\, BA^{N-1} \right] = N.$$

EXERCISE!

Here we have to get bounds on u and θ in terms of θ only. But, in principle, θ is only related with $\operatorname{div} u$. Thus, this inequality needs of some estimate of $\operatorname{curl} u$ in terms of $\operatorname{div} u$. The only possibility for this to hold is the interaction of these two components of the elastic solution u on the boundary.

Indeed, for the Cauchy problem in the whole space, in the absence of boundaries, $\operatorname{div} u$ and $\operatorname{curl} u$ satisfy, respectively:

$$(\operatorname{div} u)_{tt} - (\lambda + 2\mu)\Delta(\operatorname{div} u) - \alpha\Delta\theta = 0$$

$$\theta_t - \Delta\theta + \beta \operatorname{div} u_t = 0$$

$$(\operatorname{curl} u)_{tt} - \mu\Delta(\operatorname{curl} u) = 0.$$

In other words, in the absence of boundaries these two components, $\operatorname{div} u$ for the pressure or longitudinal waves and $\operatorname{curl} u$ for the transversal or rotational waves are fully decoupled.

IN THAT CASE IT WOULD BE IMPOSSIBLE TO GET ANY ESTIMATE ON THE TRANSVERSAL COMPONENT IN TERMS OF THE LONGITUDINAL ONE.

BUT THE BOUNDARY CONDITIONS INTRODUCE COUPLING!

COMPACT DECOUPLING

WE WANT TO REDUCE THE PROBLEM TO THE ANALYSIS OF THE LAMÉ SYSTEM. FOR THAT WE NEED TO GET RID OF THE HEAT COMPONENT.

To do that we are going to introduce a "compact decoupling" argument allowing to reduce the problem of whether

$$E(t) \leq Ce^{-\omega t}E(0), \ \forall t > 0 \tag{9}$$

holds, to a simpler question involving only the Lamé system.

Compact decoupling allows getting rid of the heat component.

The key idea: The system of the thermoelasticity is the superposition of a wave-like and a heat equation. While solutions of the wave equation do reproduce the same pattern for all t-s, those of the heat equation are very quickly dissipated. Thus, very likely, the heat equation

$$\theta_t - \Delta\theta + \beta \operatorname{div} u_t = 0$$

is, from an asymptotic point of view, very close to the elliptic one

$$-\Delta\theta + \beta \operatorname{div} u_t = 0.$$

According to this, $\nabla\theta = \beta P(u_t)$, where P is the orthogonal projection from $(L^2(\Omega))^n$ into the subspace of *curl-free* vector fields (pure gradients).

EXERCISE: SHOW THAT THE ORTHOGONAL PROJECTION FROM $(L^2(\Omega))^n$ INTO THE SUBSPACE OF PURE GRADIENTS $\{\nabla\varphi : \varphi \in H_0^1(\Omega)\}$ IS GIVEN BY THE OPERATOR P.

In one space dimension: $P\varphi = \varphi - \frac{1}{L}\int_0^L \varphi dx$.

Assuming this simplified interaction between θ and $\operatorname{div} u$ and going back to the first equation we get the decoupled system :

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \alpha\beta P u_t = 0 & \text{in} \quad \Omega\times(0,\infty) \\ \theta_t - \Delta\theta + \beta \operatorname{div} u_t = 0 & \text{in} \quad \Omega\times(0,\infty) \\ u = 0, \theta = 0 & \text{on} \quad \partial\Omega\times(0,\infty) \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega. \end{cases} \tag{10}$$

This system, which is close to the original one, is *fully decoupled*. One can solve the equation for u first, and then, knowing u, that of θ.

The compact decoupling result reads as follows:

Lemma 1 *For any* $0 < T < \infty$ *the difference of the two semigroups* $S(t) - S_d(t)$ *is compact from* H *into* $C([0,T];H)$*. In order words, for any bounded set* B *of* H *the set of trajectories*

$$\{[S(t) - S_d(t)](u^0, u^1, \theta^0) : (u^0, u^1, \theta^0) \in B\}$$

is relatively compact in $C([0,T];H)$*.*

As a consequence of it, the following holds:

Theorem 1 *Assume that* $n = 2$ *or* 3. *In the class of domains* Ω *satisfying condition* (C), *the uniform decay property for the original system of thermoelasticity holds if and only if there exists* $T > 0$ *and* $C > 0$ *such that*

$$\| \varphi^0 \|^2_{(L^2(\Omega))^n} + \| \varphi^1 \|^2_{(H^{-1}(\Omega))^n} \leq C \int_0^T \| \operatorname{div} \varphi \|^2_{H^{-1}(\Omega)} \, dt \tag{11}$$

holds for every solution of the Lamé system:

$$\begin{array}{lll} \varphi_{tt} - \mu \Delta \varphi - (\lambda + \mu) \nabla \operatorname{div} \varphi = 0 & in & \Omega \times (0, \infty) \\ \varphi = 0 & on & \partial\Omega \times (0, \infty) \ . \\ \varphi(x, 0) = \varphi^0(x), \varphi_t(x, 0) = \varphi^1(x), & in & \Omega. \end{array} \tag{12}$$

Proof: One inequality is equivalent to the other one plus a compact reminder. We can then absorb the compact remainder. This is so because the condition (C) guarantees the uniqueness property which is needed to eliminate the possible existence of a finite-dimensional non-trivial kernel, making the inequality impossible: $\operatorname{div}\varphi \equiv 0$ then $\varphi \equiv 0$.

This is the same proof as that of Poincaré's inequality by compactness and contradiction.

PROOF OF THE COMPACT DECOUPLING RESULT:

Let B be a bounded set of H. We set

$$(u(t), u_t(t), \theta(t)) = [S(t)](u^0, u^1, \theta^0),$$

$$(\tilde{u}(t), \tilde{u}_t(t), \tilde{\theta}(t)) = [S_d(t)](u^0, u^1, \theta^0)$$

and

$$(v(t), v_t(t), \eta(t)) = [S(t) - S_d(t)](u^0, u^1, \theta^0)$$

for any $(u^0, u^1, \theta^0) \in B$.

We have:

$$\begin{cases} v_{tt} - \mu\Delta v - (\lambda+\mu)\nabla \text{div} v + \alpha\nabla\eta = \alpha[\beta P\tilde{u}_t - \nabla\tilde{\theta}] & \text{in } \Omega\times(0,\infty) \\ \eta_t - \Delta\eta + \beta \text{div} v_t = 0 & \text{in } \Omega\times(0,\infty) \\ v = 0, \eta = 0 & \text{on } \partial\Omega\times(0,\infty) \\ v(0) = v_t(0) = 0, \eta(0) = 0 & \text{in } \Omega. \end{cases} \tag{13}$$

It is sufficient to check that $\beta P\tilde{u}_t - \nabla\tilde{\theta}$ is bounded in $L^1(0,T;(H^s(\Omega))^n)$ for some $s > 0$ when (u^0, u^1, θ^0) varies in B. Indeed, once this holds, then the difference (v, η) gains s derivatives and consequently it is compact in the energy space.

Let us decompose $\beta P\tilde{u}_t - \nabla\tilde{\theta}$ as follows:

$$\beta P\tilde{u}_t - \nabla\tilde{\theta} = \nabla w_1 + \nabla w_2$$

where w_1 satisfies

$$\left\{\begin{array}{ll} w_{1,t} - \Delta w_1 = 0 & \text{in } \Omega \times (0,\infty) \\ w_1 = 0 & \text{on } \partial\Omega \times (0,\infty) \\ w_1(0) = -\beta(-\Delta)^{-1}(\mathrm{div}(u^1)) - \theta^0 & \text{in } \Omega. \end{array}\right. . \tag{14}$$

and w_2 verifies

$$\left\{\begin{array}{ll} w_{2,t} - \Delta w_2 = -\beta(-\Delta)^{-1}(\mathrm{div}(\tilde{u}_{tt})) & \\ w_2 = 0 & \text{on } \partial\Omega \times (0,\infty) \\ w_2(0) = 0 & \text{in } \Omega. \end{array}\right. . \tag{15}$$

Since $\beta(-\Delta)^{-1}(\mathrm{div} u^1) + \theta^0$ is bounded in $L^2(\Omega)$, because of the regularizing effect of the heat equation, we deduce that w_1 is bounded in $L^1(0,T;H^{1+s}(\Omega))$ for any $0 < s < 1$.

On the other hand, $(-\Delta)^{-1}(\text{div}(\tilde{u}_{tt}))$ is bounded in $L^2(\Omega \times (0,T))$. Indeed,

$$\text{div}(\tilde{u}_{tt}) = (\lambda + 2\mu)\Delta\text{div}(\tilde{u}) - \alpha\beta\text{div}(\tilde{u}_t).$$

Since $\text{div}(\tilde{u}_t)$ is bounded in $L^\infty(0,T;H^{-1}(\Omega))$ it is sufficient to check that $(-\Delta)^{-1}\Delta\text{div}(\tilde{u})$ is bounded in $L^2(\Omega \times (0,T))$. This is easy to check since $\text{div}(\tilde{u})$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and, by Rellich's inequality, the trace of $\text{div}(\tilde{u})$ on $\partial\Omega \times (0,T)$ is bounded in $L^2(\partial\Omega \times (0,T))$. Therefore, w_2 is bounded in $L^2(0,T;H^1(\Omega))$ and in $L^1(0,T;H^{1+\delta}(\Omega))$ for any $0 < \delta < 1$. This concludes the proof of the decoupling result.

EXERCISE. MULTIPLYING THE LAPLACE EQUATION IN THE DIRICHLET PROBLEM BY $q(x)\cdot\nabla u$, WHERE THE VECTOR FIELD q IS AN EXTENSION OF THE NORMAL VECTOR FIELD TO THE INTERIOR, GET AN UPPER BOUND ON THE TRACE OF $\partial u/\partial\nu$ OVER $L^2(\Gamma)$.

EXTEND THIS ANALYSIS TO FINITE ENERGY SOLUTIONS OF THE WAVE EQUATION AND THE SYSTEM OF ELASTICITY.

NOTE THAT THIS ARGUMENT IS THE SAME AS POZOHAEV. IN THAT CASE THE VECTOR FIELD q IS CHOSEN TO BE $q(x) = x - x_0$.

ANALYSIS OF THE LAMÉ SYSTEM

The problem is now reduced to the analysis of the Lamé system:

$$\begin{array}{lll} \varphi_{tt} - \mu\Delta\varphi - (\lambda+\mu)\nabla\,\mathrm{div}\,\varphi = 0 & \text{in} & \Omega\times(0,\infty) \\ \varphi = 0 & \text{on} & \partial\Omega\times(0,\infty)\ . \\ \varphi(x,0) = \varphi^0(x), \varphi_t(x,0) = \varphi^1(x), & \text{in} & \Omega, \end{array} \quad (16)$$

and, more precisely, to whether the following inequality holds or not:

$$\| \varphi^0 \|^2_{(L^2(\Omega))^n} + \| \varphi^1 \|^2_{(H^{-1}(\Omega))^n} \le C \int_0^T \| \mathrm{div}\,\varphi \|^2_{H^{-1}(\Omega)}\, dt. \quad (17)$$

As mentioned above: This inequality is absolutely impossible in the whole space because $\mathrm{div}\,\varphi$ and curl φ are fully decoupled. But one may not expect it to hold for the Dirichlet problem due to the interaction of longitudinal and transversal waves on the boundary.

The velocity of propagation of longitudinal and transversal waves is as follows: $c_L = \sqrt{\lambda + 2\mu}$ and $c_T = \sqrt{\mu}$. We also set $\nu_L = 1/c_L$ and $\nu_T = 1/c_T$.

Waves are propagated along rays. The direction of the ray when reaching the boundary has two components. The tangential one will be denoted by η (it was ξ' in the previous talk).

The plane wave analysis indicates that for waves that reach the boundary almost perpendicularly or tangentially, the interaction of transversal and longitudinal waves is very weak, in the sense that one derivative is lost.

More precisely, along those rays, L^2 transversal component yield, after bouncing on the boundary, H^1 longitudinal waves.

According to this, making measurement on the longitudinal components only allows recovering the transversal one with a loss of one derivative

2-D

The Dirichlet boundary condition reads

$$\varphi_1 = \varphi_2 = 0.$$

Assume the boundary is given by the line $x_1 = 0$. Then

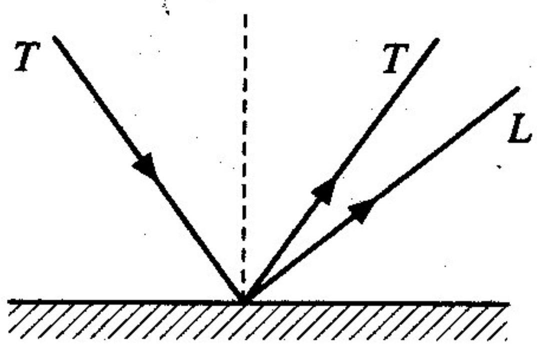
$$\partial_1 \varphi_1 = \partial_1 \varphi_2 = 0.$$

Over the boundary,

$$\text{div}\, \varphi = \partial_1 \varphi_1 + \partial_2 \varphi_2 = \partial_2 \varphi_2,$$

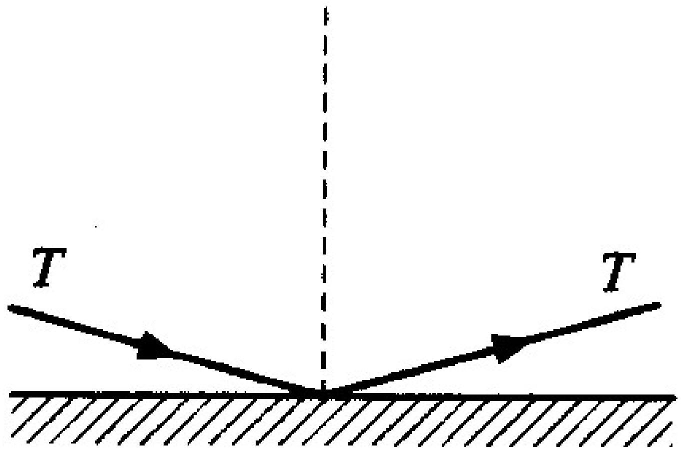
and

$$\text{curl} \varphi = \partial_2 \varphi_1 - \partial_1 \varphi_2 = \partial_2 \varphi_1.$$



$0 < |\eta| \leqq |\tau|\nu_L$

Normally, an incoming purely transversal wave produces, after bouncing, a longitudinal and a transversal wave of the same order up to a constant multiplicative factor (the transmission constant). This is the case when the tangential component of the incoming ray is small enough.



$$|\tau|\nu_L < |\eta| < |\tau|\nu_T$$

However, when the incoming wave is almost tangential, no longitudinal waves are generated after bouncing.

More precisely: One can generate a purely transversal wave in the interior of the domain. It remains purely transversal up to the time in which it reaches the interface. When this happens part of the energy is transfered into the longitudinal component. But this only affects a lower order energy, with a loss of one derivative. This makes the inequality above impossible.

AN EXAMPLE: PERPENDICULAR RAYS IN 2-D.

Assume the boundary is given by $x_1 = 0$. Consider a transversal wave that propagates in the x_1 direction, reaching the interface perpendicularly.

It has the form:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i(x_1\xi_1+\tau t)}.$$

Note that, in the present case,

* $\xi' = 0$, normal incidence.

* The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is taken as amplitude to guarantee that the incoming wave is purely transversal: $\partial_2\varphi_1 - \partial_1\varphi_2 \equiv 0$.

* To guarantee it is a transversal wave: $\tau^2 = \mu\xi_1^2$.

The reflected transversal wave wil have a similar form

$$a_T \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i(-x_1\xi_1+\tau t)},$$

a_T being the reflection coefficient of transversal waves.

The reflected longitudinal one will be of the form

$$a_L \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i(\sqrt{\mu}x_1\xi_1/\sqrt{\lambda+2\mu}+\tau t)}.$$

The overall solution will be the superposition of these three contributions:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i(x_1\xi_1+\tau t)} + a \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i(-x_1\xi_1+\tau t)} + a_L \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i(\sqrt{\mu}x_1\xi_1/\sqrt{\lambda+2\mu}+\tau t)}.$$

We now impose the Dirichlet boundary condition at $x_1 = 0$. Both components of the solution should vanish.

This implies immediately:

* $a_L = 0$

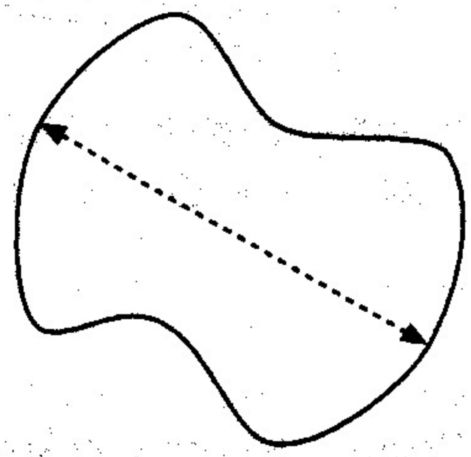
* $1 + a_T = 0$, i. e. $a_T = -1$.

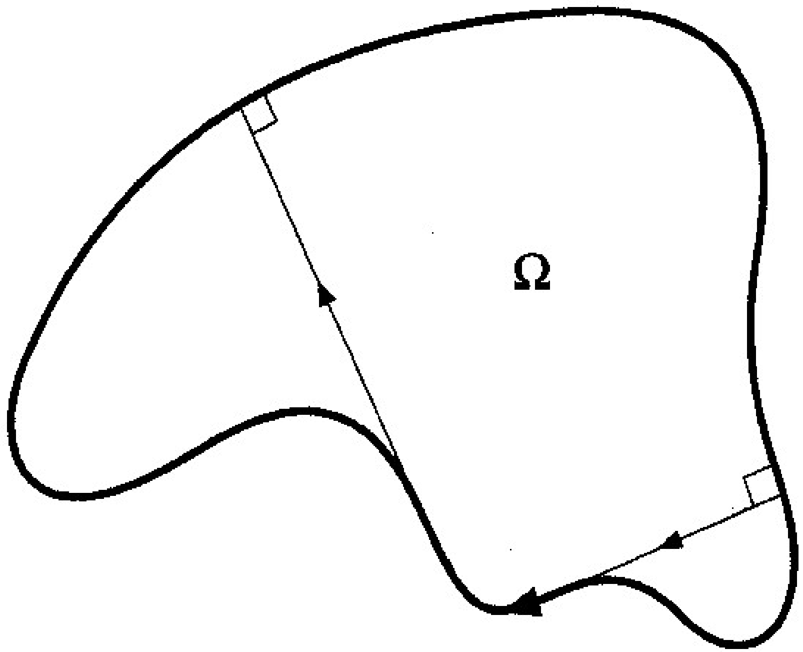
That means that the transversal wave is reflected into a purely transversal wave without contributing to the longitudinal component at all.

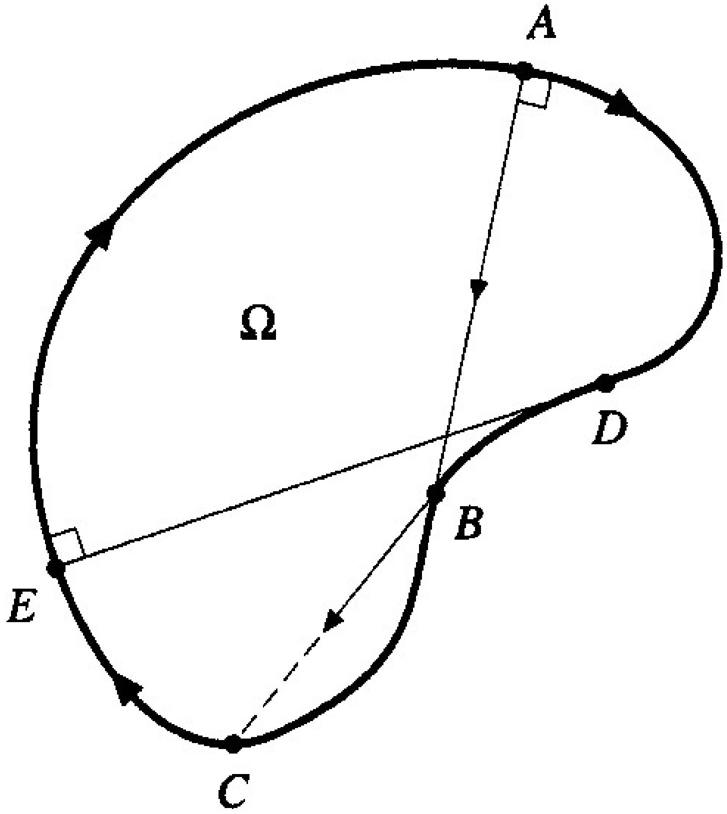
THIS IS AN EXTREME SITUATION IN WHICH A MEASUREMENT OF THE LONGITUDINAL COMPONENT DOES NOT PROVIDE ANY INFORMATION ABOUT THE TRANSVERSAL ONE.

Theorem 2 *For domains in which there is a ray of Geometric Optics of arbitrarily large length which is always reflected on the boundary perpendicularly or almost tangentially, the exponential decay property does not hold.*

This happens for instance to all convex domains. But to many others too. For those that a diameter exists, for instance.




Ω


A
Ω
D
B
E
C

According to the previous results:

- For most domains, each trajectory tends to zero. There are however pathological domains for which this does not hold.

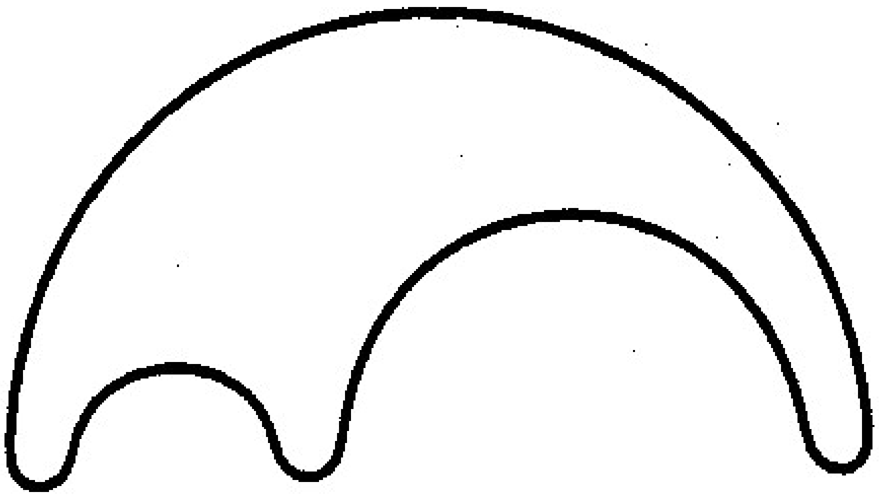
  A full classification of good and bad domains is still to be done (Pompeiu's problem).

- Even if each individual solution tends to zero, for most domains, the decay rate is not uniformly exponential. That is the case for instance for all domains which have tangent or perpendicular rays to the boundary.

ARE THERE DOMAINS WITHOUT RAYS THAT ARE ALWAYS TANGENTIAL OR PERPENDICULAR?

In those domains exponential decay holds, under assumption (C) and for suitable ranges of values of the Lamé constants, λ and μ.

Here is one:



BUT, WHAT HAPPENS IN THE MOST FREQUENT SITUATION IN WHICH (C) HOLDS, BUT THERE ARE PERPENDICULAR OR TANGENT RAYS.

We know that there is no exponential decay. What else can be said?

Theorem 3 **Inequalities with deffect.** *For most domains, if T is large enough, the following holds: If $u \in (\mathcal{D}'(\Omega \times (0,T)))^2$ solves*

$$\begin{array}{lll} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u = 0 & \textit{in} & \Omega \times (0,T), \\ u = 0 & \textit{on} & \partial\Omega \times (0,T), \end{array} . \tag{18}$$

and

$$\operatorname{div} u \in L^2(\Omega \times (0,T)) \tag{19}$$

it follows that

$$u \in L^2(\Omega \times (0,T)). \tag{20}$$

Note that this result provides a global information on u in terms of $\operatorname{div} u$. But, as the plane wave analysis predicts, there is a loss of one derivative.

As a consequence of this, the following holds:

Theorem 4 **Polynomial decay.** *For most domains, there exists* $C > 0$ *such that*

$$E(t) \leq \frac{C}{t} \parallel (u^0, u^1, \theta^0) \parallel_D^2 , \forall t > 0 \qquad (21)$$

for every solutionwith initial data in the domain

$$D = (H^2 \cap H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2 \times (H^2 \cap H_0^1(\Omega)).$$

REMARK:

In order to get polynomial decay we have chosen to work with smooth initial data, with one more L^2-derivative than finite energy solutions. This is necessary.

Indeed, there is a classical and easy result in the theory of semi-groups showing that for a dissipative semigroup, either the norm of the semigroup tends to zero or there are solutions for which the decay rate is arbitrarily slow.

In our case, we know the decay rate is not uniform. We can only expect some decay rate by restricting the class of solutions under consideration.

In addition to the energy space H, we have introduced the domain D of the generator of the semigroup $S(t)$:

$$D = (H^2 \cap H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2 \times (H^2 \cap H_0^1(\Omega)), \tag{22}$$

endowed with the natural norm. We also introduce the space H_{-1}, dual of D with respect to the pivot space H. It then follows that

$$||S(t)(u^0, u^1, \theta^0)||_D \leq ||(u^0, u^1, \theta^0)||_D, \, \forall t > 0 \tag{23}$$

for all $(u^0, u^1, \theta^0) \in D$ and also, the interpolation inequality,

$$||(u^0, u^1, u^0)||_H^2 \leq ||(u^0, u^1, \theta^0)||_D ||(u^0, u^1, \theta^0)||_{H_{-1}}, \forall (u^0, u^1, \theta^0) \in D. \tag{24}$$

As we have seen

$$||S(T)(u^0, u^1, \theta^0)||_{H_{-1}}^2 \leq C[||(u^0, u^1, \theta^0)||_H^2 - ||S(T)\left(u^0, u^1, \theta^0)||_H^2\right]. \tag{25}$$

Let us introduce the sequence of positive numbers

$$\alpha_n = \left| S(nT)(u^0, u^1, \theta^0) \right|_H^2.$$

We have

$$\alpha_n - \alpha_{n+1} \geq \frac{1}{C} \left| S((n+1)T)(u^0, u^1, \theta^0) \right|_{H_{-1}}^2$$

which combined with previous results gives

$$\alpha_n - \alpha_{n+1} \geq \frac{\alpha_{n+1}^2}{C \left| S((n+1)T)(u^0, u^1, \theta^0) \right|_D^2} \geq \frac{\alpha_{n+1}^2}{C \left| (u^0, u^1, \theta^0) \right|_D^2}, \forall n \geq 1. \tag{26}$$

Without loss of generality we may assume that $\left|(u^0, u^1, \theta^0)\right|_D = 1$.
Then,

$$\alpha_{n+1} + \frac{\alpha_{n+1}^2}{C} \leq \alpha_n, \forall n \geq 1$$

and $\alpha_1 \leq 1$.

It is then easy to see that $\alpha_n \leq \beta_n$ for all $n \geq 1$ where β_n solves

$$\beta_{n+1} + \frac{\beta_{n+1}^2}{C} = \beta_n, \forall n \geq 1$$

with $\beta_1 = 1$.

It is clear that $\beta_n \leq C'/n$, for some $C' > 0$ and all $n \geq 1$.

EXERCISE!

SIMILAR QUESTIONS ARISE FOR OTHER RELEVANT MODELS. FOR INSTANCE, IN THE CONTEXT OF MAGNETOELASTICITY THE SAME IDEAS ALLOW REDUCING THE PROBLEM OF THE LARGE TIME BEHAVIOR TO THAT OF ANALYZING WHTHER THERE ARE SOLUTIONS OF THE LAMÉ SYSTEM FOR WHICH $\varphi_n \equiv 0$.

Are there eigenfunctions of the Lamé system for which φ_n vanishes?

* In 2-D it is easy to build polygonal domains for which those objects exist;

* But, in 3-D, there are no such domains in the class of Lipschitz domains.

The main reason:

* In 1-d there are eigenfunctions of the laplacian taking the value 1 on the boundary and such that its minimum is ≥ -1:

psi(x)=sin (x).

* In 2-d these objects do not exist. This is a consequence of the moving plane argument.

EXERCISE: Given a bounded domain Ω in $\mathbf{R}^2$. Show that if

$$-\Delta\psi = \sigma\psi \qquad \text{in } \Omega;$$

$$\psi = 1 \qquad \text{on } \partial\Omega.$$

Then,

* either $\psi = 1$;

or

* $\min_\Omega \psi < -1$