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Open problems and future directions of research

Enrique Zuazua
Universidad Autónoma
28049 Madrid, Spain
enrique.zuazua@uam.es
http://www.uam.es/enrique.zuazua

- **The heat equation**

$$u_t = \Delta u \qquad \text{heat equation}.$$

Its semi-discrete version:

$$u_t = \Delta_h u. \quad \vec{u} = \sum_{k=1}^{N} a_k e^{-\lambda_k^h t} \vec{w}_k^h.$$

Spurious high frequency solutions are this time:

$$\vec{u} = e^{-\lambda_N^h t} \vec{w}_N^h.$$

They are exponentially damped out.

Once again, the controls are bounded and converge, as $h \to 0$, to the controls of the limiting heat equation.

But this is only true in $1 - d$. As for the Schrödinger, the example of the eigenvector localized in the diagonal shows that the control properties of the semi-discrete heat equation in the square are not as good for the contrinuous one.

It would be necessary to develop the semi-discrete or the discrete version of the Carleman inequalities that have been systematically employed to derive observability properties for continuous heat equations.

Note that, very likely, they will only yield unique continuation and observability results within suitable classes of filtered solutions.

- **Numerical viscosity.**

Are there numerical schemes, dissipating conveniently the high frequency spurious oscillations, for which the observability inequalities become uniform on the discrete parameter h?

$$\vec{u}_h'' + A_h \vec{u}_h + h^\alpha A_h \vec{u}_h' = 0.$$

Compare with

$$u_{tt} - \Delta u - \epsilon \Delta u_t = 0.$$

At the level of stabilization it is well known that $\alpha = 2$ to prove the exponential decay.

At the level of exact controllability the only known results are due to S. Micu, 2006.* They are valid for $\alpha = 1$ and in $1 - d$. Note that, for controllability to hold, $\alpha = 1$ is optimal. In particular, the uniform controllability property fails to hold for the exponent $\alpha = 2$ which is the optimal one for stabilization.

The extension to the multi-dimensional case is a completely open problem.

*S. Micu, Uniform boundary controllability of a semi-discrete $1 - D$ wave equation with vanishing viscosity, to appear.

- **Initial data with a finite number of Fourier components.**

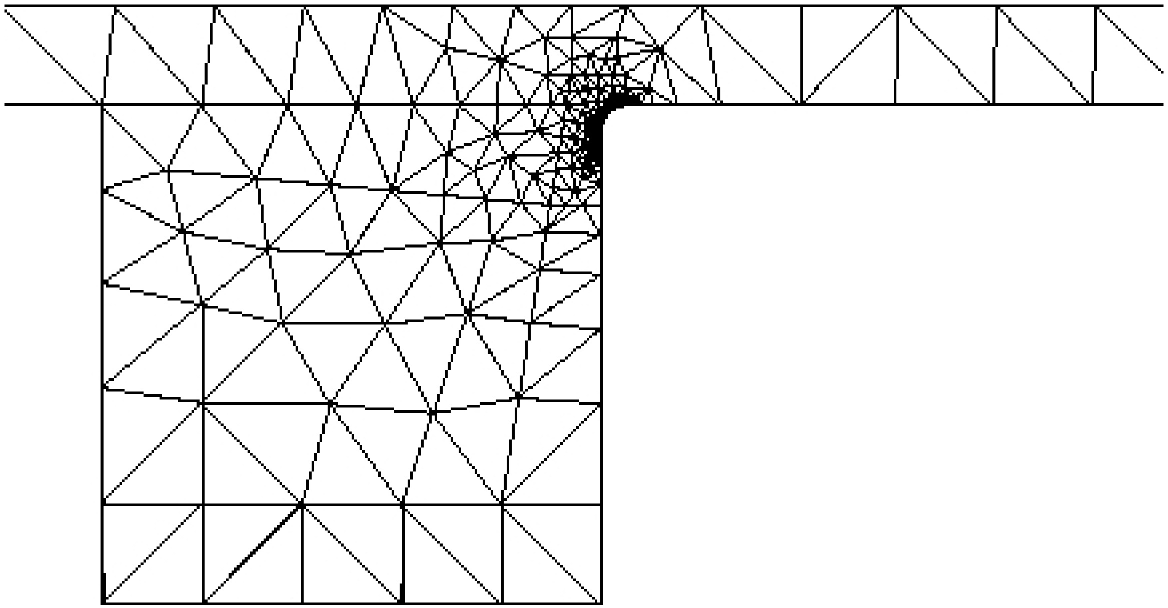
  It is also known that, in $1-d$, when the initial data has only a finite number of Fourier components the controls do converge (S. Micu, 2002)[†].

  Is this true in several space dimensions?

  No tool seems to be available to address this problem.

[†] S. Micu, Uniform boundary controllability of a semi-discrete 1-D wave equation, Numer. Math., 91 (2002), pp. 723–768.

- Complex geometries, variable and irregular coefficients, irregular meshes, the system of elasticity, ...



- **Mixed Finite Elements**

  Design multi-dimensional mixed finite elements with appropriate dispersion diagram so that all numerical waves propagate with a velocity independent of h.

- **Nonlinear state equations.**

  Very little is known about possible extensions to non-linear equations.

  At the PDE level the best tool to deal with such problems is combining fixed point arguments with fine estimates for linear equations with potentials that are normally obtained by means of Carleman inequalities, a topic that is to be developed at the discrete level.

- **Inverse Problems**

  The observability inequalities that have been derived for control can also be used for solving a number of inverse problems.

  Most of the ideas we have developed can be applied in thjis context too:

  a) The fact that identifiability is not uniform for numerical schemes as the mesh-size parameter tends to zero;

  b) The fact that filtering restablishes the uniformity of the identifiability property,....

  But a complete theory is still to be developed.

- **Shape and optimal design**

  Very little is known about shape and optimal design problems for controllability and stabilization. This is true in the continuous setting and also in what concerns the convergence of numerical optimal shapes to the continuous ones.

  We refer to the results by D. Chenais and E. Z.[‡] for the Dirichlet $2-d$ elliptic optimal design by means of finite elements.

[‡] D. Chenais & E. Z. Finite Element Approximation of 2D Elliptic Optimal Design, JMPA, 85 (2006), 225-249.

- **Transparent boundary conditions and PML**

  Most of the analysis we have developed here can be adapted to analyze the tgrue behavior of numerical transparent boundary conditions and perfectly matching layers.

  This problem has been addressed by S. Ervedoza E: Z. (2006) in the $1-d$ case but a complete theory is still to be developed.

- **Wave, heat and schrödinger equations on graphs and networks.**

  There is a quite complete theory about the control of the wave equation on $1-d$ networks. The numerical analysis of this issue along the lines discussed in these lectures is completely open.

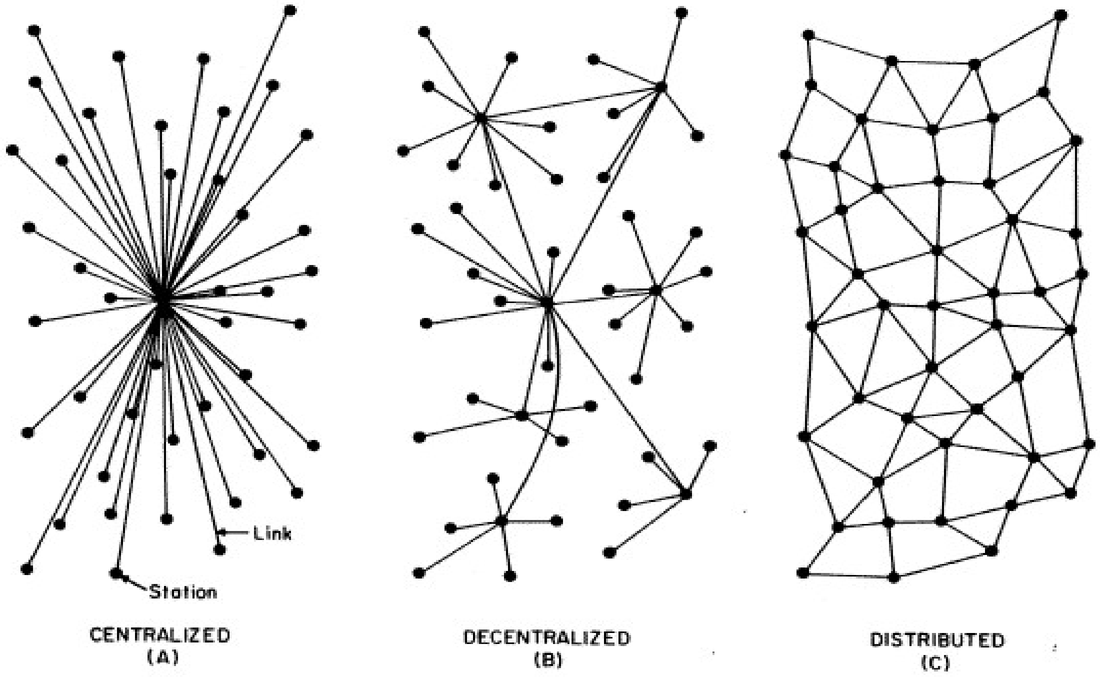


FIG. I — Centralized, Decentralized and Distributed Networks

R. DAGER & E. Z. Wave propagation and control in $1-d$ vibrating multi-structures. Springer Verlag. "Mathématiques et Applications", Paris. 2005

- **Dispersive numerical schemes**

  Similar ideas can be applied for designing convergent numerical algorithms for the non linear Schrödinger equation that mimic the same Strichartz inequalities as the continuous one:

  L. IGNAT & E. Z., Dispersive Properties of Numerical Schemes for Nonlinear Schrödinger Equations, Proceedings of FoCM'2005, Santander, June-July 2005.

Further references:

- Survey articles on the control of PDE:

  E. Z. "Controllability and Observability of Partial Differential Equations: Some results and open problems", in *Handbook of Differential Equations: Evolutionary Equations, vol. 3*, C. M. Dafermos and E. Feireisl eds., Elsevier Science, pp. 527-621.

  E. Z. Controllability of Partial Differential Equations and its Semi-Discrete Approximation. Discrete and Continuous Dynamical Systems, 8 (2) (2002), 469-513.

  E. Z. Some Problems and Results on the controllability of Partial Differential Equations. Progress in Mathematics, vol. 169, 1998, Birkhäuser Verlag Basel/Switzerland, 276–311.