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Some open problems in PDE control

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1.- Optimal design for waves: a microlocal approach

2.- Sharp observability estimates for heat equations

3.- Robust control of linear finite-dimensional systems

4.- Control of Kolmogorov's equation

1.- Optimal design for waves: a microlocal approach

Internal stabilization of waves: Let ω be an open subset of Ω. Consider:

$$\begin{cases} y_{tt} - \Delta y = -y_t 1_\omega & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega, \end{cases}$$

where 1_ω stands for the characteristic function of the subset ω.

The energy dissipation law is then

$$\frac{dE(t)}{dt} = -\int_\omega |y_t|^2 dx.$$

Question: Do they exist $C > 0$ and $\gamma > 0$ such that

$$E(t) \leq Ce^{-\gamma t} E(0), \quad \forall t \geq 0,$$

for all solution of the dissipative system?

This is equivalent to an observability property: There exists $C > 0$ and $T > 0$ such that

$$E(0) \leq C \int_0^T \int_\omega |y_t|^2 dx dt.$$

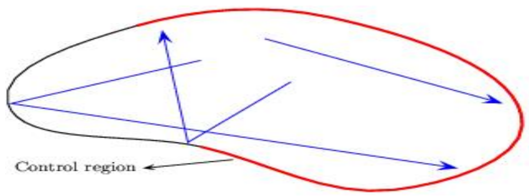
In other words, the exponential decay property is equivalent to showing that the dissipated energy within an interval $[0, T]$ contains a fraction of the initial energy, uniformly for all solutions.

This estimate, together with the energy dissipation law, shows that

$$E(T) \leq \sigma E(0)$$

with $0 < \sigma < 1$. Accordingly the semigroup map $S(T)$ is a strict contraction. By the semigroup property one deduces immediately the exponential decay rate.

The observability inequality and, accordingly, the exponential decay property holds if and only if the support of the dissipative mechanism, Γ_0 or ω, satisfies the so called the Geometric Control Condition (GCC) (Ralston, Rauch-Taylor, Bardos-Lebeau-Rauch,...)



Rays propagating inside the domain Ω following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case. The proof requires tools from Microlocal Analysis.

Given a subdomain ω (or Γ_0) for which the stabilization problem holds, it is natural to address the problem of optimizing the profile of the damping potential $a = a(x)$ to enhance the exponential decay rate. Consider

$$\begin{cases} y_{tt} - \Delta y = -a(x) y_t 1_\omega & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega. \end{cases}$$

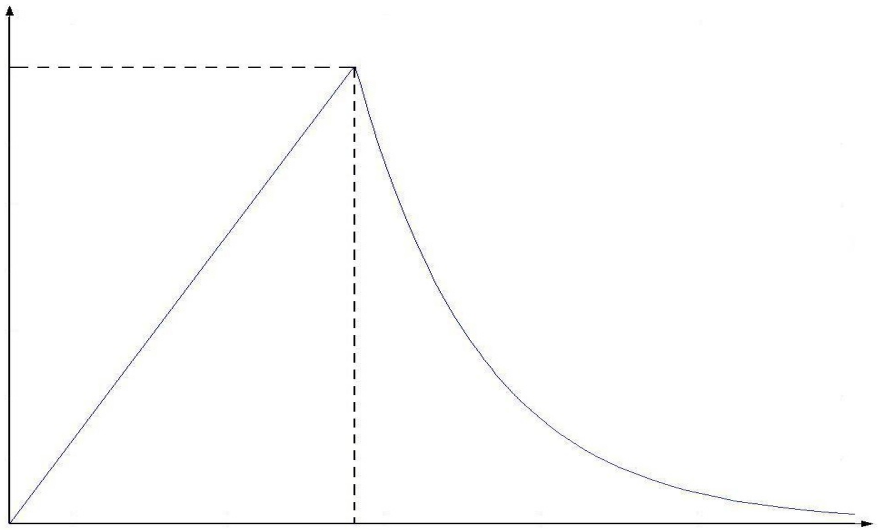
Then, for any $a > 0$ the exponential decay property holds:

$$E(t) \leq C e^{-\gamma_a t} E(0), \quad \forall t \geq 0.$$

Obviously, the decay rate γ_a depends on the damping potential a.

It is therefore natural to analyze the nature of the mapping $a \to \gamma_a$. One could also analyze the dependence of the decay rate on the geometry of the subdomain ω (γ_a depends also on ω).

Against the very first intuition this map is not monotonic. A $1-d$ spectral computation yields:



Some known results:

- $1-d$: The exponential decay rate coincides with the spectral abscissa within the class of BV damping potentials. For large eigenvalues $Re(\lambda) \sim -\int_\omega a(x)dx/2$ (S. Cox & E. Z., CPDE, 1993). Thus:

$$\gamma_a \leq \int_\omega a(x)dx.$$

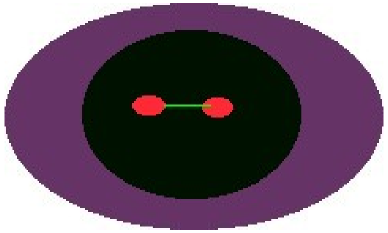
  But there is another limitation due to **overdamping**. Despite of this, the following surprising result was proved (Castro-Cox, SICON, 2001): The decay rate may be made arbitrarily large by approximating singular potentials of the form $a(x) = 2/x$ for the space interval $\Omega = (0, 1)$.

- In the multidimensional case the situation is even more complex. In this case it is not longer true that the spectral absicssa characterizes the exponential decay rate. There are actually two quantities that enter in such characterization (G. Lebeau, 1996):

  - The spectral abscissa;

  - The minimum asymptotic average (as $T \to \infty$) of the damping potential along rays of Geometric Optics.

    The later is in agreement with our intuition of waves traveling wlong rays of Geometric Optics.



This is a typical situation in which the spectral abscissa does not suffice to capture the decay rate. The damping mechanism is active on the outer neighborhood of the exterior boundary. When the domain is the ellipsoid this produces the exponential decay. But, in the presence of the two holes, the exponential decay rate is lost, due to the existence of a trapped ray that never meets the damping region. In this case the decay rate is zero but the spectrum is not essentially affected if the holes are small enough. Thus the spectrum is unable to characterize the null decay rate.

The optimal design of the damping potential with constraints (size, shape, etc.) is still widely open.

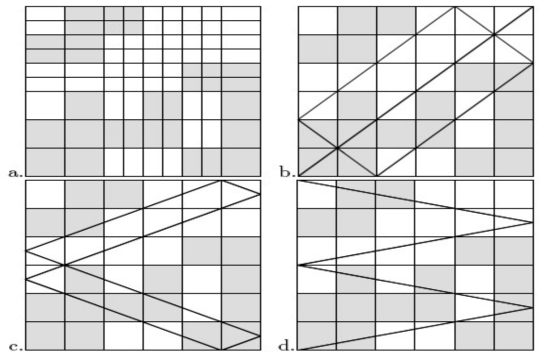
– Hébrard-Henrot, SCL, 2003. They show the complexity of the problem in the $1-d$ case for small amplitude damping potentials located on the union of a finite number of intervals.

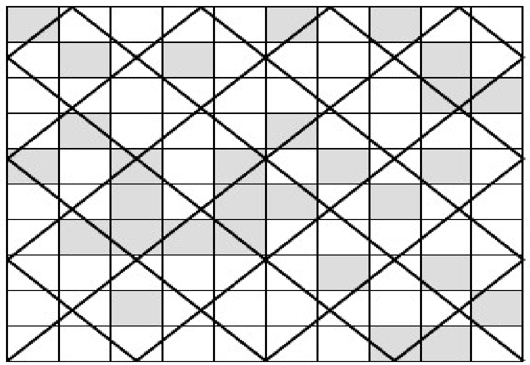
– Hébrard-Humbert, 2003: Optimization of the shape of ω in a square domain in view of the geometric optics quantity entering in the characterization of the decay rate.

– Cox-Henrot, Ammari-Tucsnak, 2002: $1-d$ problems with damping terms located at a single point through a Dirac delta. Eigenvalues are complex valued, and they depend both on the amplitude of the damping and the diophantine properties of the point support.

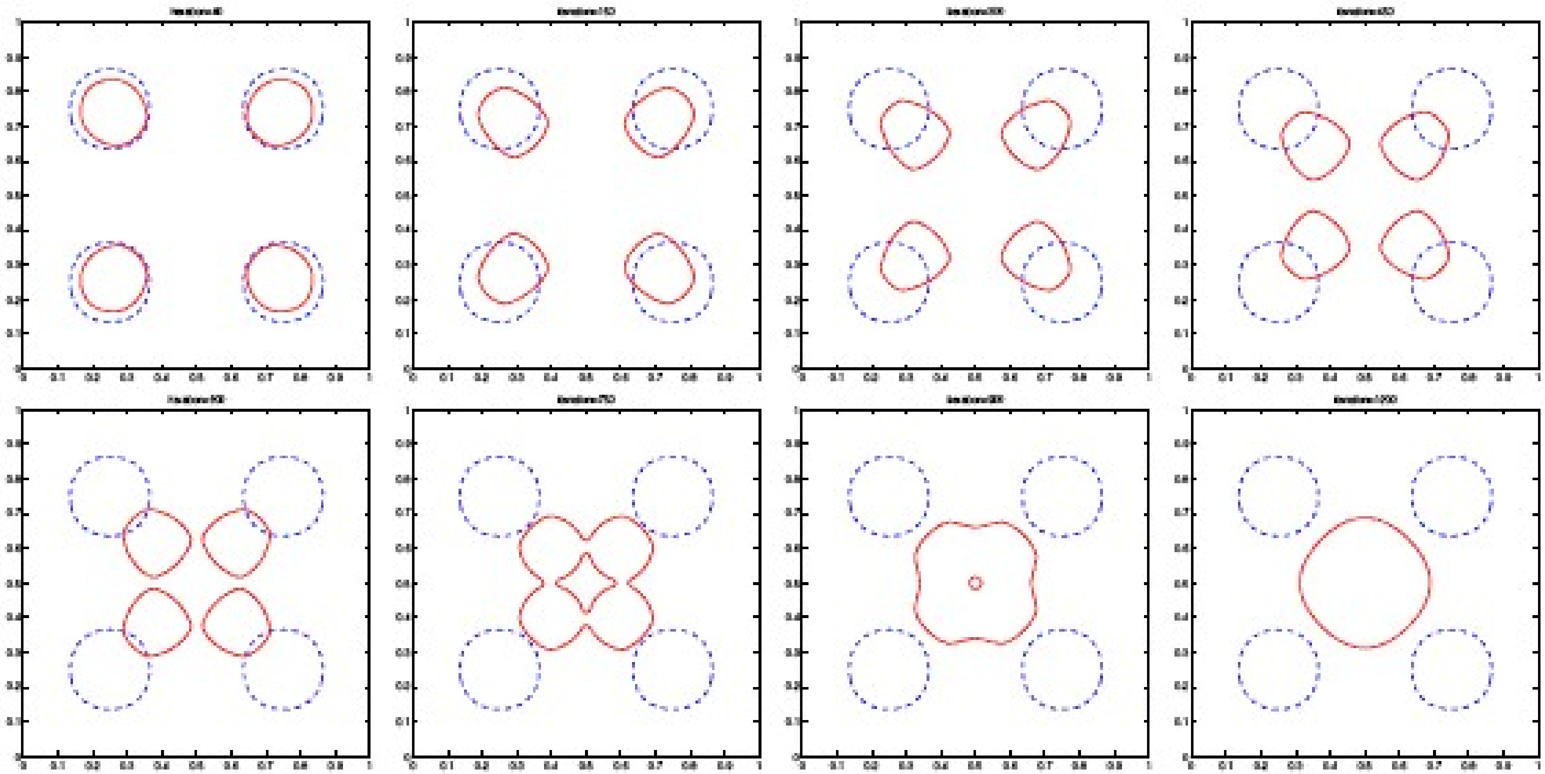
– A. Münch, P. Pedregal, F. Periago 2005, ...: Young measures, relaxation, Level set methods.

– And many others...





Hébrard-Humbert, 2003



A. Munch, 2005.

The main difficulties are related to the fact that there is no variational principle characterizing the decay rate, and to the complex way in which the eigenvalues depend on the damping potentials, and the different way they do it for low/high/intermediate frequencies, for small/large amplitudes of the damping potentials, with respect to the shape of the support, ...

Futhermore, not always all authors deal with the same problem. For instance, the optimal damping for a given initial datum may differ significantly from the optimal damping when considering globally all possible solutions...

This is the case even for constant damping potentials k. The optimal damping for the ℓ-th eigenfunction is $k = 2\sqrt{\mu_\ell}$.

Open problem # 1.1: Characterize the optimal dampers for given initial data. How do they depend on their regularity? What about initial data with a finite number of Fourier components?

Open problem # 1.2: Given the subdomain ω, characterize the optimal damping potential for all finite energy solutions.

Open problem # 1.3: Given a damping potential (constant, for instance), characterize the optimal subdomain ω for its location.

Open problem # 1.4: Optimal dampers for the billiard. What is the subdomain that absorbs faster all rays? How this depends on the geometry of Ω? How it depends on the number of connected components of ω and on its size? What about variable coefficients/metrics?

2.- Sharp observability estimates for heat equations

THE CONTROL PROBLEM

Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x, 0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

1_ω denotes the characteristic function of the subset ω of Ω where the control is active.

We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (1) admits an unique solution

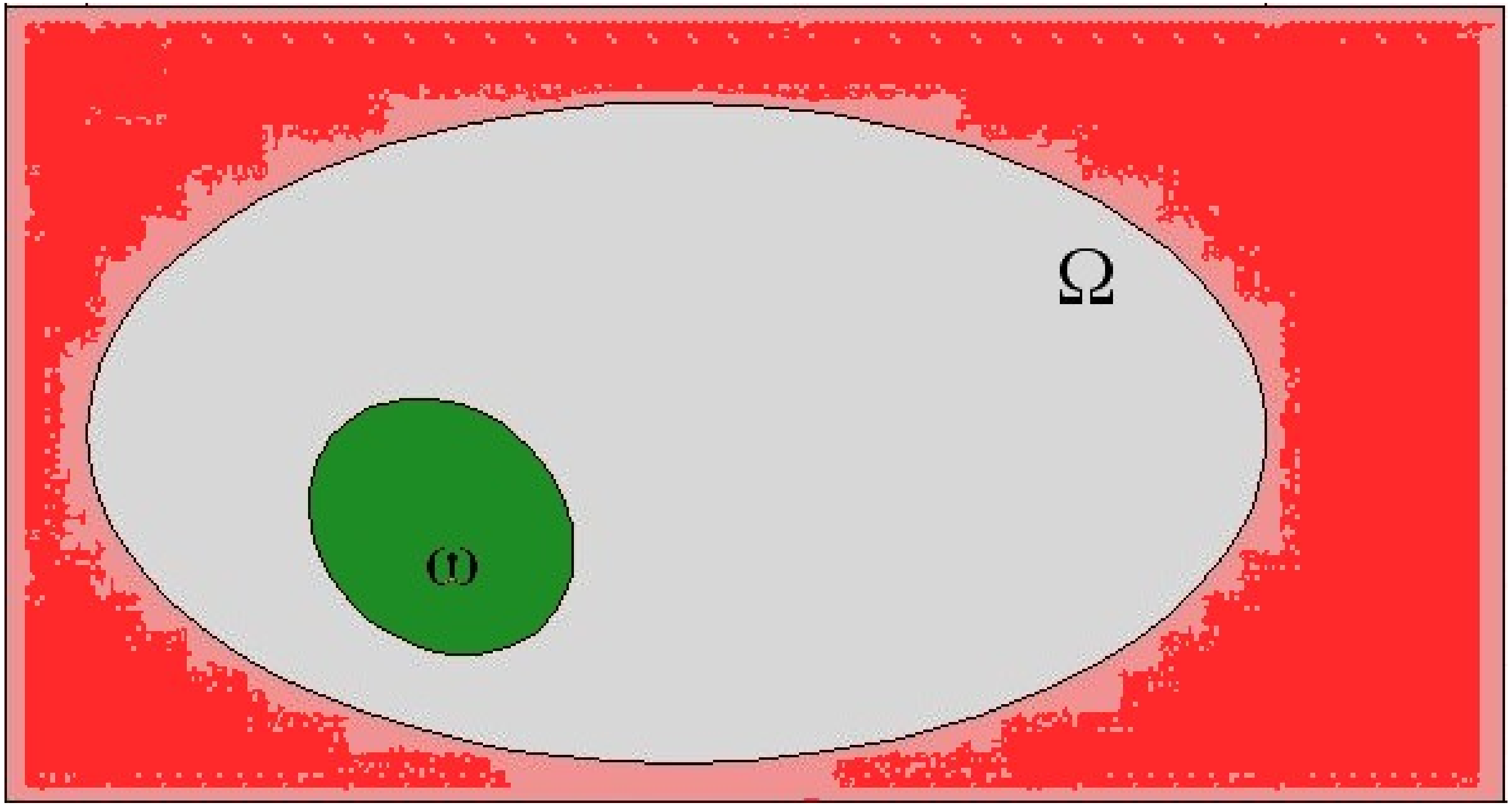
$$u \in C\left([0, T]\,; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

$$u = u(x,t) = solution = state, \ f = f(x,t) = control$$

Well known result (Fursikov-Imanuvilov, Lebeau-Robbiano,...) : The system is null-controllable in any time T and from any open non-empty subset ω of Ω.

In other words, for all $u_0 \in L^2(\Omega)$ there exists a control $f \in L^2(\omega \times (0,T))$ such that the corresponding solution satisfies

$$u(T) \equiv 0.$$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The control of minimal L^2-norm can be found by minimizing

$$J_0(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0)u^0 dx \tag{2}$$

Obviously, the functional is continuous and convex from $L^2(\Omega)$ to $\mathbb{R}$ and coercive because of the observability estimate:

$$\| \varphi(0) \|_{L^2(\Omega)}^2 \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{3}$$

This estimate was proved by Fursikov and Imanuvilov (1996) using Carleman inequalities. In fact the same proof applies for equations with smooth (C^1) variable coefficients in the principal part and for heat equations with lower order potentials.

Consider the heat equation or system:

$$\begin{cases} \varphi_t - \Delta \varphi = 0 & \text{in } Q \\ \varphi = 0 & \text{on } \Sigma \\ \varphi(0,x) = \varphi^0(x) & \text{in } \Omega \end{cases} \tag{4}$$

where φ takes values in $\mathbb{R}^N$.

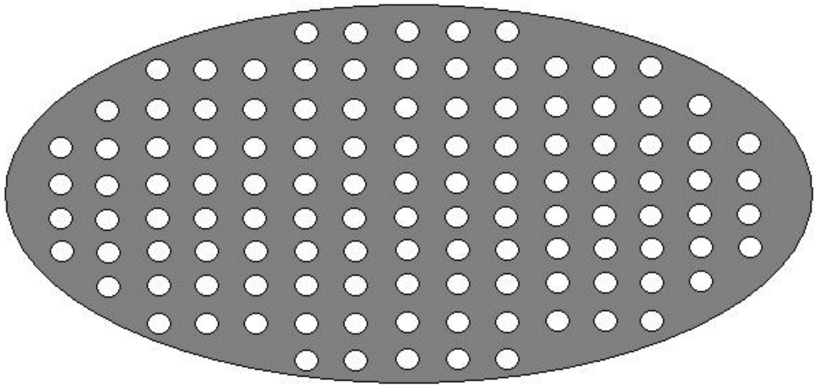
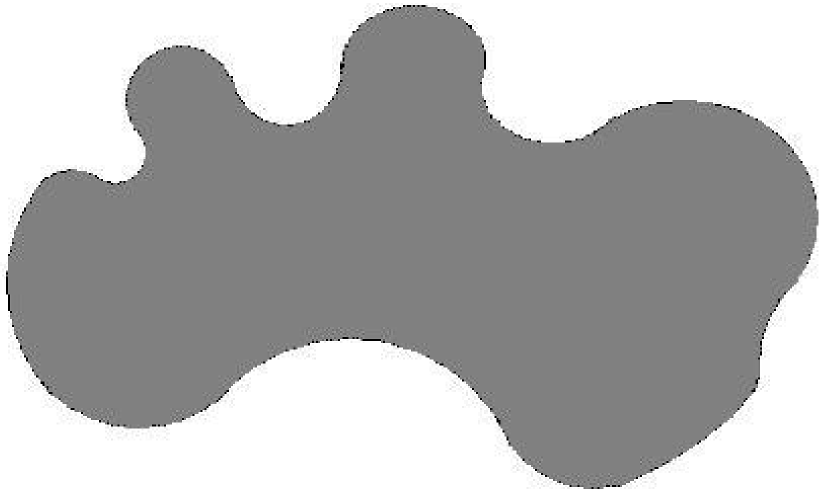
In the absence of potential, the Carleman inequality yields the following observability estimate for the solutions of the heat equation:

$$\int_0^\infty \int_\Omega e^{\frac{-A}{t}} \varphi^2 dx dt \leq C \int_0^\infty \int_\omega \varphi^2 dx dt.$$

Open problem # 2.1: Characterize the best constant A in this inequality:

$$A = A(\Omega, \omega).$$

The Carleman inequality approach allows establishing some upper bounds on A depending on the properties of the weight function. But this does not give a clear path towards the obtention of a sharp constant.

L. Miller (2003) , by inspection of the heat kernel, proved

$$A > \ell^2/4$$

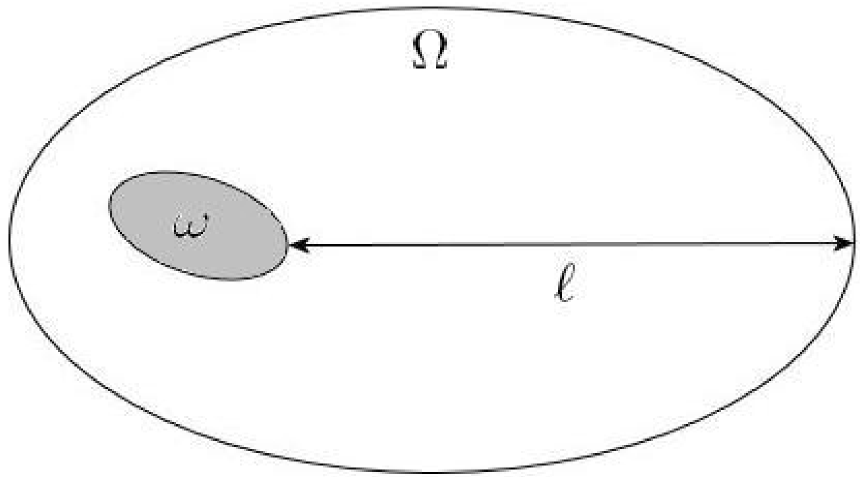
where ℓ is the length of the largest geodesic in $\Omega \setminus \omega$.

Recall that:

$$G(x,t) = (4\pi t)^{-n/2} \exp\left(\frac{-|x|^2}{4t}\right).$$

then, the following upper bound holds for the Green function in Ω:

$$G_\Omega(x,y,t) \le C t^{-n/2} \exp\left(\frac{-d^2(x,y)}{(4+\delta)t}\right).$$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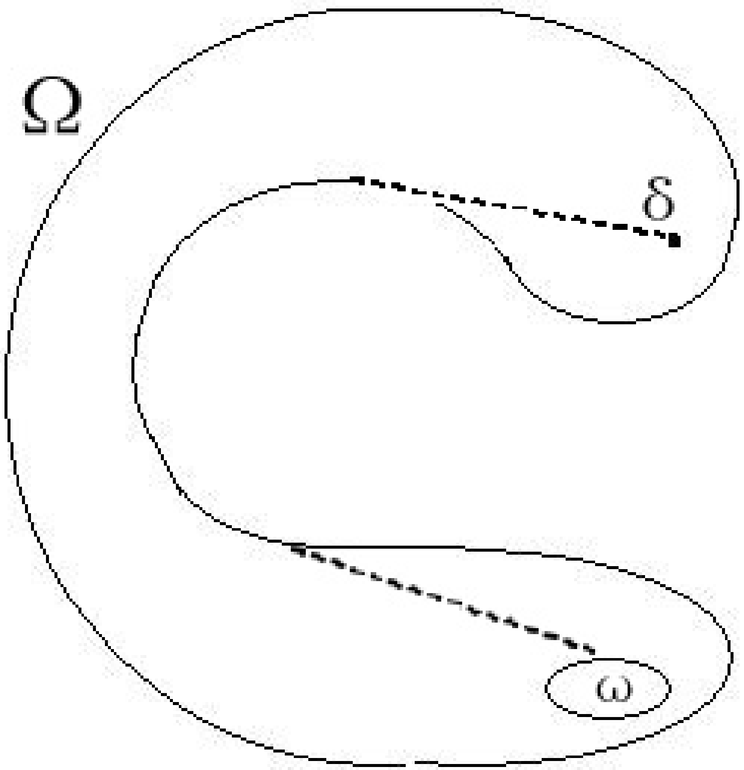
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The spectral approach

Lebeau and Robbiano proposed (1996) a spectral proof of the null controllability that, by duality, yields observability inequalities too. The key ingredient is the following estimate on the linear independence of restrictions of eigenfunctions of the laplacian:

Theorem 1 *(*Lebeau + Robbiano, 1996)

Let Ω be a bounded domain of class C^∞. For any non-empty open subset ω of Ω there exist $B, C > 0$ such that

$$Ce^{-B\sqrt{\mu}} \sum_{\lambda_j \leq \mu} \mid a_j \mid^2 \leq \int_\omega \left| \sum_{\lambda_j \leq \mu} a_j \psi_j(x) \right|^2 dx \qquad (5)$$

for all $\left\{a_j\right\} \in \ell^2$ and for all $\mu > 0$.

Open problem # 2.2: To characterize the best constant $B = B(\Omega, \omega)$.? Is the constant B in this spectral inequality related to the best constant $A > 0$ in the parabolic one?

Several works have also been devoted to get upper bounds on the best constant A using Carleman inequalities, Kannai's tranform and the control of waves under the so-called Geometric Control Condition (GCC) (Miller),one-dimensional tools from non-harmonic Fourier series, moment problems and number theory (Seidman; Tucsnak and Tenenbaum,...) but, as far as we know, even in $1-d$ the problem of getting sharp upper bounds in open.

Open problem # 2.3: In one space-dimension is it true that

$$A = \ell^2/4 \quad ???$$

Open problem # 2.4: Possible connections with well known results on decay rates for damped wave equations in which both microlocal quantities and spectral ones enter, that only coincide in $1-d$ (see Section #1)???

3.- Robust control of linear finite-dimensional systems

Partially dissipative linear hyperbolic systems

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} A_j \frac{\partial w}{\partial x_j} = -Bw \ , \ x \in \mathbb{R}^m \ , \ w \in \mathbb{R}^n \qquad (6)$$

$$\begin{array}{c} A_1, ..., A_m \\ \text{symmetric} \end{array} \ \Bigg| \ B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \begin{array}{l} \updownarrow n_1 \\ \updownarrow n_2 \end{array} \qquad \begin{array}{c} X^t D X > 0 \\ \forall X \in \mathbb{R}^{n_2} - \{0\} \end{array}$$

Goal: Understand the asymptotic behavior as $t \to \infty$.

Apply Fourier transform:

$$\frac{\partial \widehat{w}}{\partial t} = (-B - iA(\xi))\widehat{w} \quad \text{where} \quad A(\xi) := \sum_{j=1}^{m} \xi_j A_j$$

Lack coercivity : $\langle[B+iA(\xi)]X, X\rangle = \langle BX, X\rangle = \langle DX_2, X_2\rangle \not\geqslant c|X|^2$

is compatible with the decay depending on ξ:

$$\exp[(-B - iA(\xi))t] \leqslant Ce^{-\lambda(\xi)t}$$

PARTIALLY DISSIPATIVE LINEAR HYPERBOLIC SYSTEM

$$\equiv$$

m-PARAMETER (ξ) FAMILY OF FINITE-DIMENSIONAL PARTIALLY DISSIPATIVE n-DIMENSIONAL SYSTEMS.

The asymptotic behavior of solutions is determined by the behavior of the function $\xi \to \lambda(\xi)$ giving the decay rate as a function of ξ.

A quantitative measure of the decay rate as a function of ξ:

$A_1, ..., A_m$ symmetric	$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \begin{matrix} \updownarrow n_1 \\ \updownarrow n_2 \end{matrix}$	$A(\xi) := \sum_{j=1}^{m} \xi_j A_j$

$\xi = \rho\omega \in \mathbb{R}^m \quad \rho > 0 \quad \omega \in S^{m-1} \quad (m_k) \uparrow$ well chosen

$$N_{*,\epsilon}(\omega) := \min\{\sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2; x \in S^{n-1}\}.$$

Theorem 2 *(K. Beauchard and E. Z.)*

$\exists \epsilon_* > 0, c > 0$ *such that* $\forall \epsilon \in (0, \epsilon_*)$,

$$\exp[(-B - i\rho A(\omega))t] \leqslant 2e^{-cN_{*,\epsilon}(\xi)\min\{1,\rho^2\}t}.$$

Remark : **(SK)= (Shizuta-Kawashima)** $\Leftrightarrow$ **Kalman rank condition for** $(A, B) \Leftrightarrow N_{*,\epsilon}(\omega) \geqslant N_{*,\epsilon} > 0, \forall \omega \in S^{m-1}$.

In general, $N_{*,\epsilon}(\omega)$ may vanish for some values of $\omega \in S^{m-1}$, in which case the decomposition of solutions and its asymptotic form is more complex.

The set of degeneracy :

$$\mathcal{D}(B+iA(\xi)) = \{\xi \in \mathbb{R}^m; \text{rank}[B|BA(\xi)|...|BA(\xi)^{n-1}] < n\}$$

is an algebraic submanifold

- either $|\mathcal{D}| = 0 \Leftrightarrow N_{*,\epsilon} > 0$ a.e. $\Rightarrow$ strong L^2 stability;

  or

- $\mathcal{D} = \mathbb{R}^m$: $\exists$ non dissipated solutions

Open problem # 3.1: Characterize and classify, in terms of (A, B), the possible sets of degeneracy $\mathcal{D}$.

Open problem # 3.2: Characterize and classify, in terms of (A, B), the possible degenerate behaviors of $N_{*,\epsilon}(\omega)$ as $\omega \to \mathcal{D}$.

Open problem # 3.3: Classify the possible asymptotic behaviors of partially dissipative hyperbolic systems as $t \to \infty$.

Open problem # 3.4: Describe the controllability properties of m-parameter families of finite-dimensional systems:

$$x'(t) + iA(\xi)x(t) = Bu(t) \quad \text{where} \quad A(\xi) := \sum_{j=1}^{m} \xi_j A_j.$$

An example:

Theorem 3 *(K. Beauchard & E. Z.)*

When $n_1 = 1$, $\mathcal{D}$ *is a vector subspace of* $\mathbb{R}^m$ *and*

$$N_{*,\epsilon}(\omega) \geqslant c \min\{1, dist(\omega, \mathcal{D})^2\}, \forall \omega \in S^{m-1}.$$

Example: $n = m = 2$; $\mathcal{D} = \{(\xi_1, \xi_2) : a_{21}^1 \xi_1 + a_{21}^2 \xi_2 = 0\}$.

4.- Control of Kolmogorov's equation

Null control of the Kolmogorov equation:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial^2 f}{\partial v^2} = u(t,x,v)1_\omega(x,v), (x,v) \in \mathbf{R}_x \times \mathbf{R}_v, t \in (0,+\infty). \tag{7}$$

In a recent work with K. Beauchard, we consider the particular case where where $\omega = \mathbf{R}_x \times \Big[\mathbf{R}_v - [a,b]\Big]$.

Equivalently, one may address the following observability inequality for the adjoint system:

$$\begin{cases} \frac{\partial g}{\partial t} - v\frac{\partial g}{\partial x} - \frac{\partial^2 g}{\partial v^2} = 0, (x,v) \in \mathbf{R}_x \times \mathbf{R}_v, t \in (0,T), \\ g(0,x,v) = g_0(x,v), (x,v) \in \mathbf{R}_x \times \mathbf{R}_v. \end{cases} \tag{8}$$

$$\int_{\mathbf{R}_x \times \mathbf{R}_v} |g(T,x,v)|^2 dxdv \leqslant C \int_0^T \int_\omega |g(t,x,v)|^2 dxdvdt.$$

Theorem 4 *(K. Beauchard and E. Z.)*

In the particular case where $\omega = \mathbf{R}_x \times \left[\mathbf{R}_v - [a,b]\right]$ *the observability inequality holds for the adjoint system and the Kolmogorov equation is null controllable.*

Ideas of the proof:

– Fourier transform in v:

$$\begin{cases} \frac{\partial \widehat{f}}{\partial t}(t,\xi,v) + i\xi v \widehat{f}(t,\xi,v) - \frac{\partial^2 \widehat{f}}{\partial v^2}(t,\xi,v) = \widehat{u}(t,\xi,v) 1_{\mathbb{R}-[a,b]}(v), \\ \widehat{f}(0,\xi,v) = \widehat{f}_0(\xi,v). \end{cases} \tag{9}$$

– Decay:

$$\left|\hat{f}(t,\xi,.)\right|_{L^2(\mathbb{R})} \leqslant \left|\hat{f}_0(\xi,.)\right|_{L^2(\mathbb{R})} e^{-\xi^2 t^3/12}, \forall \xi \in \mathbb{R}, \forall t \in \mathbb{R}_+. \quad (10)$$

– Control depending on the parameter ξ with cost

$$e^{C(T)\max\{1,\sqrt{|\xi|}\}}.$$

The exponentially large cost of control for high frequencies is compensated by the exponential (and stronger) decay rate.

Open problem # 4.1: Similar results hold for other geometries of control sets?

Open problem # 4.2: What about more general classes of hypoelliptic equations?

Open problem # 4.3: May Carleman inequalities be applied directly on the Kolmogorov system without using Fourier transform?

Open problem # 4.4: How are related the notions of hypoellipticity and hypocoercivity with the property of null controllability (connections with Open Problems #2.X on the heat kernel).