DyCon: Dynamic Control

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$$\text{DyCon} = \text{DYNAMICS} + \text{CONTROL}$$

Control = Cybernetics (Norbert Wiener (1894–1964) : The science of control and communication in animals and machines.

Dynamics = Dynamical Systems: Mathematical model describing time evolution (continuous or discrete) of a quantity of interest (state) on a phase space.

Institutions, October 2016-September 2021

DeustoTech (Chair of Computational Mathematics)
Technological Center of the University of Deusto, Bilbao, Basque Country, Spain

Mainly oriented to research and technological transfer in the fields of Informatics, Computer and Data Sciences and Computational Mathematics

Universidad Autónoma de Madrid (UAM), Spain
In particular through its Doctoral Program in Mathematics

Scientific context and objectives

We develop an innovative research agenda, at the meeting point of Applied Mathematics and Scientific Computing with the following goals :

- To contribute with **new key mathematical methods** and results
- To develop the corresponding **numerical methodology**
- To build **new computational tools and software**
- **Technological transfer** to bridge the gap to **applications**:
  - Electrical networks
  - Aeronautics
  - Management of natural resources
  - Material sciences
  - Collective behaviour
  - Economics, urban design
  - Biomedicine

Work Packages

- WP1: Control of parameter dependent problems (PDC).
- WP2: Long time horizon control and the turnpike property (LTHC).
- WP3: Control under constraints (CC).
- WP4: Inverse design and control in the presence of singularities (SINV).
- WP5: Models with memory and hybrid systems (MHM).
- WP6: From finite to infinite-dimensional models (FI).
- WP7: DYCON-COMP Computational Platform (DYCON-COMP).

Control of parameter dependent problems (PDC)

In applications, **models are not completely known**, since the relevant parameters (deterministic or stochastic) are subject to **uncertainty**.

It is therefore essential to develop **robust analytical and computational methods** to deal with parameter-dependent families of systems in a stable and computationally efficient way.

Greedy strategies will be developed.

Long time horizon control (LTHC)

Control problems are most often considered in finite time intervals, without paying attention to how the time horizon affects the optimal strategy. However, the **effective available time horizon** is one of the critical factors: the design of medical therapies, sonic boom minimisation for supersonic aircrafts, etc.

This is linked to the classical concept of **turnpike**, due to von Neumann and Samuelson (Nobel Prize 1970) in game theory.

According to it, if the uncontrolled free dynamics tends to a steady configuration, **optimal dynamic control strategies are nearly steady** as well.

This leads to a significant **reduction of the computational cost** of optimal strategies.

Control under constraints (CC)

Constraints, often formulated as unilateral bounds on the controlled state, play a fundamental role in many applications, such as those related to diffusion processes (heat conduction, mathematical biology and population dynamics, etc.).

Most of the existing theory of controllability for PDEs has been developed in the absence of constraints on the states. Thus, in practice, **most of the available controllability results do not ensure that controlled trajectories fulfil the physical constraints** of the process under consideration.

Inverse design and control in the presence of singularities (SINV)

Some important PDE models represent a major challenge from a control viewpoint for two (closely related) reasons:

- Solutions lack regularity properties and develop **singularities** in finite time, making linearization methods inapplicable.
- The property of **backward uniqueness** is lost and the most elementary control problem (but relevant in applications), that of **inverse design** (initial source identification), is severely ill-posed.

General theoretical and computational methods to tackle these issues do not exist.

Models with memory and hybrid systems (MHM)

Relevant models may involve **memory** effects. Then, controls are required **to anticipate the future effect of past memory accumulated along the controlled trajectory**.

The effect of memory can be, in some cases, handled by augmenting the dynamical system under consideration, viewing the memory terms as further state variables.

This leads to a new class of systems of **hybrid** nature, involving, for instance, Partial and Ordinary Differential Equations (PDE + ODE).

This new viewpoint allows to understand the limitations of standard control strategies and to develop new ones to handle memory effects.

From finite to infinite-dimensional models (FI)

The **interplay between finite and infinite-dimensional dynamics** in control arises in different ways in applications such as **collective dynamics** and **pedestrian flow** or **material sciences**.

Control techniques are very sensitive to the dimension of the system and need to be carefully tuned to ensure stability as the dimension grows to reach continuous infinite-dimensional mean-field models.

Effective models are often described by continuous PDEs involving non-local (in space) terms, modelling **interactions between agents**.

DYCON-COMP Computational Platform (DYCON-COMP)

The research conducted in the previous WPs will lead to the development of **algorithms of minimal computational cost** that will be integrated in the **DYCON Computational Platform DYCON-COMP** that will be made freely available on the DYCON web.

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Averaged control

Consider the transport equation with unknown velocity v,

$$f_t + v f_x = 0,$$

and take averages with respect to v. Then

$$g(x,t) = \int f(x,t;v)\rho(v)dv$$

and, for the Gaussian density ρ:

$$\rho(v) = (4\pi)^{-1/2}\exp(-v^2/4)$$

the averaged dynamics becones

$$g(x,t) = h(x,t^2); \quad h_t - h_{xx} = 0.$$

Dramatic change of type in the PDE under consideration.

One can then employ parabolic techniques to control averages. [1] [2]

[1] E. Z., Automatica, 2014.

[2] Q. Lü, E. Z., J. Math. Pures Appl. 105 (2016) 367-414.

Greedy algorithms

But averaged control gives very little information on how efficiently each specific realisation of the system is actually controlled.

Greedy algorithms have been developed in

M. Lazar & E. Z., Greedy controllability of finite dimensional linear systems, Automatica, 74 (2016) 327-340.

ensuring that the most relevant snapshots of the parameters are found to approximate optimally the manifolds (parameter-dependent one) of controls in an optimal way in the sense of Kolmogorov width.

The subject is still mainly to be developed. So far the greedy choice depends on the initial data to be controlled

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[3] M. Choulli E. Z., Lipschitz dependence of the coefficients on the resolvent and greedy approximation for scalar elliptic problems, C. R. Acad. Sci. Paris, Ser I 354 (2016) 1174-1187.

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A real life experiment/performance

Feedback control of a shep-flock by the action of a dog, guided by the shepherd

We develop and control a **guidance by repulsion** model based on the two-agents framework: *the driver*, which tries to drive the *evader*. [4]

1. The driver thus follows the evader but cannot be arbitrarily close to it (because of chemical reactions, animal conflict, etc).
2. The evader moves away from the driver but doesn't try to escape beyond a not so large distance.
3. The driver is of course faster than the evader.
4. At a critical short distance, the driver can display a **circumvention maneuver** around the evader that forces the evader to change the direction of its motion.
5. Thus, by adjusting the onset and offset of the circumvention maneuver, the evader can be driven towards a desired target or along a given trajectory.

[4] R. Escobedo, A. Ibanez and E.Zuazua, Optimal strategies for driving a mobile agent in a "guidance by repulsion" model, Communications in Nonlinear Science and Numerical Simulation, 39 (2016), 58-72.

One shep + one dog

$$
\begin{aligned}
\dot{\vec{u}}_d(t) &= \vec{v}_d(t), \\
\dot{\vec{u}}_e(t) &= \vec{v}_e(t), \\
m_d \dot{\vec{v}}_d(t) &= -A \left(1 - \frac{\delta_c^2}{\left| \vec{u}_d(t) - \vec{u}_e(t) \right|^2} \right) \frac{\vec{u}_d(t) - \vec{u}_e(t)}{\left| \vec{u}_d(t) - \vec{u}_e(t) \right|^2} \\
&\qquad -\kappa(t) \frac{(\vec{u}_d - \vec{u}_e)^{\perp}}{\left| \vec{u}_e(t) - \vec{u}_d(t) \right|} - \nu_d \vec{v}_d(t), \\
m_e \dot{\vec{v}}_e(t) &= B \frac{\vec{u}_e(t) - \vec{u}_d(t)}{\left| \vec{u}_e(t) - \vec{u}_d(t) \right|^2} - \nu_e \vec{v}_e(t), \\
\vec{u}_d(t_0) &= \vec{u}_d^0, \quad \vec{u}_e(t_0) = \vec{u}_e^0, \quad \vec{v}_d(t_0) = 0 \quad \text{and} \quad \vec{v}_e(t_0) = 0.
\end{aligned}
$$

A second scenario: $k(t)$ is chosen by the shepherd in a feedback manner, in view of the alignment of the flock-dog-flock

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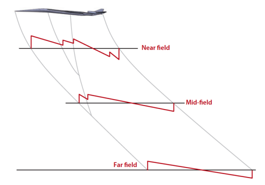
Sonic boom

- Goal: the development of supersonic aircrafts, sufficiently quiet to be allowed to fly supersonically over land.
- The pressure signature created by the aircraft must be such that, when reaching ground, (a) it can barely be perceived by humans, and (b) it results in admissible disturbances to man-made structures.



Juan J. Alonso and Michael R. Colonno, Multidisciplinary Optimization with Applications to Sonic-Boom Minimization, Annu. Rev. Fluid Mech. 2012, 44:505 – 526.

Many other examples in biomedicine, social sciences, economics, lead to natural questions of control in long time.

Sustainable growth is a long-term challenge.

And two key issues arise:

- Develop specific tools for long time control horizons.
- Build numerical schemes capable of reproducing accurately the control dynamics in long time intervals.

Geometric/Symplectic integration

Numerical integration of the pendulum (A. Marica)[5]

[height=5cm]2DibujoEZ4025

[5]HAIRER, E., LUBICH, Ch., WANNER, G.. Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations. 2nd ed. Berlin : Springer, 2006, 644 p.

Joint work with L. Ignat & A. Pozo, Math of Computation, 2015

Consider the 1-D conservation law with or without viscosity:

$$u_t + \left[u^2\right]_x = \varepsilon u_{xx}, \quad x \in \mathbb{R}, t > 0.$$

Then[6] :

- If $\varepsilon = 0$, $u(\cdot, t) \sim N(\cdot, t)$ as $t \to \infty$;
- If $\varepsilon > 0$, $u(\cdot, t) \sim u_M(\cdot, t)$ as $t \to \infty$,

u_M is the constant sign self-similar solution of the viscous Burgers equation (defined by the mass M of u_0), while N is the so-called hyperbolic N-wave.

[6] Y. J. Kim & A. E. Tzavaras, *Diffusive N-Waves and Metastability in the Burgers Equation*, SIAM J. Math. Anal. **33**(3) (2001), 607–633.

Lack of commutativity

$$\lim_{t \to \infty} \lim_{\varepsilon \to 0} \neq \lim_{\varepsilon \to 0} \lim_{t \to \infty}$$

Conservative schemes

Let us consider now numerical approximation schemes for the inviscid problem ($\varepsilon = 0$):

$$\begin{cases} u_j^{n+1} = u_n^j - \dfrac{\Delta t}{\Delta x}\left(g_{j+1/2}^n - g_{j-1/2}^n\right), & j \in \mathbf{Z}, \mathbf{n} > \mathbf{0}. \\ u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx, & j \in \mathbf{Z}. \end{cases}$$

The approximated solution u_Δ is given by

$$u_\Delta(t,x) = u_j^n, \quad x_{j-1/2} < x < x_{j+1/2},\, t_n \leq t < t_{n+1},$$

where $t_n = n\Delta t$ and $x_{j+1/2} = (j + \frac{1}{2})\Delta x$.

Is the large tine dynamics of these discrete systems, a discrete version of the continuous one?

3-point conservative schemes

1. Lax-Friedrichs

$$g^{LF}(u,v) = \frac{u^2+v^2}{4} - \frac{\Delta x}{\Delta t}\left(\frac{v-u}{2}\right),$$

2. Engquist-Osher

$$g^{EO}(u,v) = \frac{u(u+|u|)}{4} + \frac{v(v-|v|)}{4},$$

3. Godunov

$$g^{G}(u,v) = \begin{cases} \min_{w\in[u,v]} \frac{w^2}{2}, & \text{if } u \le v, \\ \max_{w\in[v,u]} \frac{w^2}{2}, & \text{if } v \le u. \end{cases}$$

Asymptotic correctness as $t \to \infty$?

- All these methods converge in the classical sense of numerical analysis.
- This refers to convergence in finite time intervals $[0, T]$!!!
- But do they behave correctly as $t \to \infty$?
- Note that, computationally, roughly, you can choose Δx and Δt, but once you do this, you have to rely on what simulations give as $t \to \infty$.

Numerical viscosity

We can rewrite three-point monotone schemes in the form

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{(u_{j+1}^n)^2-(u_{j-1}^n)^2}{4\Delta x}=R(u_j^n,u_{j+1}^n)-R(u_{j-1}^n,u_j^n)$$

where the numerical viscosity R can be defined in a unique manner as

$$R(u,v)=\frac{Q(u,v)(v-u)}{2}=\frac{\lambda}{2}\Big(\frac{u^2}{2}+\frac{v^2}{2}-2g(u,v)\Big).$$

For instance:

$$R^{LF}(u,v)=\frac{v-u}{2\Delta t},$$

$$R^{EO}(u,v)=\frac{\lambda}{4}(v|v|-u|u|),$$

$$R^G(u,v)=\begin{cases}\frac{\lambda}{4}\mathrm{sign}(|u|-|v|)(v^2-u^2), & v\le 0\le u,\\ \frac{\lambda}{4}(v|v|-u|u|), & \text{elsewhere.}\end{cases}$$

Theorem (L. Ignat, A. Pozo & E. Z.; Lax-Friedrichs scheme)

Consider $u_0 \in L^1(\mathbf{R})$ and Δx and Δt such that $\lambda \left| u^n \right|_{\infty,\Delta} \leq 1$, $\lambda = \Delta t/\Delta x$. Then, for any $p \in [1, \infty)$, the numerical solution u_Δ given by the Lax-Friedrichs scheme satisfies

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{p})} \left| u_\Delta(t) - w(t) \right|_{L^p(\mathbb{R})} = 0,$$

where the profile $w = w_{M_\Delta}$ is the unique solution of

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = \frac{(\Delta x)^2}{2\Delta t} w_{xx}, & x \in \mathbf{R}, t > 0, \\ w(0) = M_\Delta \delta_0, \end{cases}$$

with $M_\Delta = \int_{\mathbb{R}} u^0_\Delta$.

Theorem (L. Ignat, A. Pozo & E. Z.; Engquist-Osher and Godunov schemes)

Consider $u_0 \in L^1(\mathbf{R})$ and Δx and Δt such that $\lambda \left| u^n \right|_{\infty,\Delta} \leq 1$, $\lambda = \Delta t/\Delta x$. Then, for any $p \in [1,\infty)$, the numerical solutions u_Δ given by Engquist-Osher and Godunov schemes satisfy the same asymptotic behavior but for the hyperbolic $N - wave$ $w = w_{p_\Delta, q_\Delta}$ unique solution of

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = 0, \quad x \in \mathbf{R}, t > 0, \\ w(0) = M_\Delta \delta_0, \quad \lim_{t\to 0} \int_0^x w(t,z)dz = \begin{cases} 0, & x < 0, \\ -p_\Delta, & x = 0, \\ q_\Delta - p_\Delta, & x > 0, \end{cases} \end{cases}$$

with $M_\Delta = \int_{\mathbb{R}} u_\Delta^0$ and
$p_\Delta = -\min_{x\in\mathbb{R}} \int_{-\infty}^x u_\Delta^0(z)dz$ and $q_\Delta = \max_{x\in\mathbb{R}} \int_x^\infty u_\Delta^0(z)dz$.

Why?

- For the Lax-Friedrichs scheme diffusion is linear, it is invariant with respect to time:

$$-\frac{(\Delta x)^2}{2\Delta t} w_{xx}$$

as $t \to \infty$.

- But, for the Engquist-Osher and Godunov schemes the viscosity is non-linear of the order, roughly, of $-(u^2)_{xx}$. Taking into account that we have the uniform a priori bound

$$|u(x,t)| \leq Ct^{-1/2}$$

they amount of viscosity $\to 0$ as $t \to \infty$.

Nonlinear numerical viscosity is better. It self-adapts to solutions' behaviour.

Need of choosing the right numerical scheme to reproduce the right dynamics.

Example: Forward dynamics

[height=6cm]2nwave025

Example: Inverse design by means of EO scheme and gradient descent (A. Pozo)

Example: Inverse design in frontogenesis (M. Morales)

Comments

1. Numerical schemes that are asymptotically wrong tend to yield incorrect solutions to control problems.
2. This happens in long time intervals but also in finite time horizons: increasing the number of iterations of a gradient descent method top gain accuracy amounts to run the state equation many times, thus reproducing the long time behaviour.
3. Asymptotic correctness is also the key to reproduce the relevant property of *turnpike* for control problems in long-time horizons.

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1. DyCon offers the opportunity to those researchers, specially young trainees, to get involved in a challenging research program in the interface between Applied Mathematics, Computational Sciences and their interactions.
2. We would be delighted to run cooperative work.

Gracias por la amable invitación del ICMAT y vuestra atención.

http://enzuazua.net/dycon/