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# Hardy inequalities, heat kernels and wave propagation

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## Motivation & Goal

- Motivation:

  PDE with singular potentials arising in combustion theory and quantum mechanics.

- Goal:

  Revise the existing theory of well-posedness, asymptotic behavior, control, etc. when replacing $-\Delta$ by $-\Delta - \frac{\lambda}{|x|^2}$ both in the elliptic and in the evolution context.

Part of the literature on singular elliptic and parabolic problems:


- S. Chandrasekhar, An introduction to the study of stellar structure, New York, Dover, 1957.
- I. M. Gelfand, Some problems in the theory of quasilinear equations, Amer. Math. Soc. Transl., **29** (1963), 295-381.
- J. Serrin, *Pathological solution of an elliptic differential equation*, Ann. Scuola Norm. Sup. Pisa, **17** (1964), 385–387.
- D. S. Joseph & T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rat. Mech. Anal., **49** (1973), 241-269.
- F. Mignot, F. Murat, J.-P. Puel, Variation d'un point de retournement par rapport au domaine, Comm. P. D. E. **4** (1979), 1263-1297.
- P. Baras, J. Goldstein, *The heat equation with a singular potential,* Trans. Amer. Math. Soc. **284** (1984), 121–139.
- T. Gallouet, F. Mignot & J. P. Puel, Quelques résultats sur le problème $-\Delta u = \lambda e^u$. C. R. Acad. Sci. Paris Sér. I, Math. **307** (7) (1988), 289-292.

## Examples:

Example 1:

$$-\Delta u - \mu(1+u)^p = 0,$$

$$p > n/(n-2),\ \mu = \frac{2}{p-1}(n - \frac{2p}{p-1}).$$

$$u(x) = |x|^{-2/(p-1)} - 1$$

After "linearization":

$$-\Delta v - \frac{\lambda}{|x|^2} v = f.$$

with

$$\lambda = \frac{2p}{p-1}(n - \frac{2p}{p-1}).$$

Example 2:

$$-\Delta u - \lambda e^{u} = 0, \; \lambda = 2(N-2)$$

$$u(x) = -2\log(|x|).$$

After "linearization":

$$-\Delta v - \frac{\lambda}{|x|^2} v = f.$$

Elliptic PDE's with exponential nonlinearities also appear in models for drift-diffusion.

Example 3:

$$-\Delta u = |\nabla u|^q, \; u(x) = c_q(|x|^{-(2-q)/(q-1)} - 1)$$

Linearization:

$$-\Delta v = q|\nabla u|^{q-2}\nabla u \cdot \nabla v \sim -\Delta v = \mu \frac{1}{|x|^2} x \cdot \nabla v.$$

This type of singular problem, with singularities in the first order term, can be treated similarly as the previous ones. This is seen easily when analyzing its coercivity since

$$\int \frac{1}{|x|^2} x \cdot \nabla v \, v dx = \frac{1}{2} \int \frac{1}{|x|^2} x \cdot \nabla (v^2) dx = \frac{N-2}{2} \int \frac{v^2}{|x|^2} dx.$$

Warning! Linearization is formal in these examples.

Solutions are too singular to allow performing a true linearization in the corresponding energy spaces.

On the other hand, the complex behavior of solutions with respect the parameter $\lambda$ shows that Inverse Function Theorem fails to apply because of the lack of an appropriate functional setting.

This is an issue that needs and deserves further analysis and clarification.

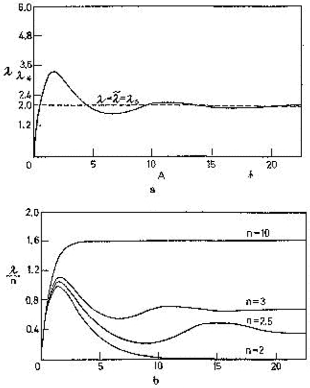



Fig. 1a. Bifurcation diagram for solutions of problem (II.3) with $n=3$. This curve was computed by numerical integration of (II.3). It gives the values of $\lambda=\lambda(A)$ for which solutions of (II.3) or the equivalent problem (I.2) are possible. For a fixed $\lambda$ it is possible to have different solutions $u(r,\lambda)$ having different values of $u(0,\lambda)=A$. When $\lambda>\lambda_* \approx 3.35$ there are no solutions. When $\lambda=\lambda_s=\tilde{\lambda}=2$ there are infinitely many solutions having different values of $A$.

Fig. 1b. Bifurcation diagrams for the solutions of problem (II.3). This figure is constructed by numerical integration of (II.3) for different values of $n$. When $n=2$ and $\lambda \neq \lambda_*$ there are either two solutions ($\lambda<\lambda_s$) or no solutions $\lambda>\lambda^*$. When $2<n<10$ there are infinitely many solutions for $\lambda=\tilde{\lambda}=\lambda_s$. When $n \gtrless 10$ and $\lambda<\lambda_*=\lambda_s$ the solutions are unique.

D. Joseph et al., 1973.

## The Cauchy problem

$$\begin{cases} u_t - \Delta u - \frac{\lambda}{|x|^2} u = 0 & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases}$$

Baras-Goldstein (1984), $N \geq 3$:

- Global existence for $\lambda \leq \lambda_* = (N-2)^2/4$;
- Instantaneous blow-up if $\lambda > \lambda_* = (N-2)^2/4$.

Explanation: Hardy's inequality:

$$\lambda_* \int_\Omega \frac{\varphi^2}{|x|^2} dx \leq \int_\Omega |\nabla \varphi|^2 dx.$$

Optimal not achieved constant: $\varphi = |x|^{-(N-2)/2}$.

Warning! In dimension $N = 2$ this inequality fails....$\lambda_* = 0$ (see Tintarev's talk).

## Preliminaries on Hardy inequalities:

The classical Hardy inequality ensures that

$$\frac{(N-2)^2}{4}\int_{\mathbf{R}^N}\frac{\varphi^2}{|x|^2}dx \leq \int_{\mathbf{R}^N}|\nabla\varphi|^2 dx.$$

The proof is easy:

$$\frac{\varphi(x)}{|x|} = -\int_1^\infty \frac{x}{|x|}\cdot\nabla\varphi(tx)dt.$$

And apply the Minkowski inequality in $L^2(\mathbf{R}^N)$.

- Of course, it also holds in $H_0^1(\Omega)$ for any domain $\Omega$.
- It guarantees the coercivity of the operator $-\Delta - \lambda/|x|^2$ in $H_0^1(\Omega)$, for $\lambda < \lambda_* = (N-2)^2/4$.

## Hardy-Poincaré inequality

But this inequality fails to yield coercivity for the critical value $\lambda_* = (N-2)^2/4$.

H. Brézis-J. L. Vázquez, 1997:

$$\lambda_* \int_\Omega \frac{\varphi^2}{|x|^2} dx + C(\Omega) \int_\Omega \varphi^2 dx \le \int_\Omega |\nabla \varphi|^2 dx, \ \forall \varphi \in H_0^1(\Omega).$$

Later improved [1]: $0 < s < 1$,

$$\lambda_* \int_\Omega \frac{\varphi^2}{|x|^2} dx + C(\Omega) ||\varphi||_s^2 \le \int_\Omega |\nabla \varphi|^2 dx, \ \forall \varphi \in H_0^1(\Omega).$$

$-\Delta - \frac{\lambda_*}{|x|^2} I$ is almost as coercive as $-\Delta$.

[1] J. L. Vázquez & E. Z. The Hardy inequality and the asymptotic behavior of the heat equation with an inverse square potential. J. Funct. Anal., 173 (2000), 103–153.

For the critical value $\lambda_*$, the elliptic operator $-\Delta - \frac{\lambda_*}{|x|^2} I$ plays the role of $-\Delta$ but in the slightly larger space $\mathcal{H}(\Omega)$, the closure of $\mathcal{D}(\Omega)$ with respect to the norm

$$||\varphi||_{\mathcal{H}} = \left[ \int_\Omega \left[ |\nabla \varphi|^2 - \lambda_* \int_\Omega \frac{\varphi^2}{|x|^2} \right] dx \right]^{1/2}.$$

The elliptic and parabolic/hyperbolic theories are then the same by replacing $H_0^1(\Omega)$ by $\mathcal{H}(\Omega)$.

But note that this only happens in the bounded domain case since the Poincaré remainder term can not catched up in the whole space $\mathbf{R}^N$.

## Idea of the proof:

$$\Omega = B(0,1); \quad \varphi = \varphi(r) \to \psi(r) = r^{(N-2)/2}\varphi(r).$$

$$||\varphi||_{\mathcal{H}} = \left[ \int_0^1 |\varphi'(r)|^2 r\, dr \right]^{1/2}.$$

Over the space of radially symmetric functions

$$-\Delta - \frac{\lambda_*}{|x|^2} I \quad \text{ in } \mathbf{R}^3 \sim -\Delta \ \text{ in } \mathbf{R}^2.$$

This guarantees coercivity in $H^s$, for $0 < s < 1$.

When $\lambda > \lambda_*$ this transformation yields

$$-\psi'' - \frac{\psi'}{r} - c\frac{\psi}{r^2} = f,$$

with $c > 0$. Consequently we have a non-admissible perturbation of the 2-d Laplacian. The equation does not make sense in the context of distributions....

## The Dirichlet problem for the parabolic operator: Three cases.

$$\begin{cases} u_t - \Delta u - \frac{\lambda}{|x|^2} u = 0 & \text{in} \quad \Omega \times (0, \infty) \\ u = 0 & \text{on} \quad \Gamma \times (0, \infty) \\ u(x, 0) = u^0(x) \in L^2(\Omega) & \text{in} \quad \Omega. \end{cases}$$

- $0 < \lambda < \lambda_*$: u$\in C\left([0, \infty)] ; L^2(\Omega)\right) \cap L^2\left(0, \infty; H_0^1(\Omega)\right)$.
- $\lambda = \lambda_*$: u$\in C\left([0, \infty)] ; L^2(\Omega)\right) \cap L^2(\Omega)\left(0, \infty; \mathcal{H}(\Omega)\right)$.
- $\lambda > \lambda_*$: Lack of well-posedness.

Furthermore, in the first two cases the $L^2$-norm of solutions decays exponentially.

Solutions have to be interpreted in the semigroup sense.

Uniqueness does not hold in the distributional one. For instance, for

$$\lambda = \lambda_*, \; u(x) = |x|^{-(N-2)/2} \log(1/|x|),$$

is a singular stationary solution. It is not the semigroup solution.

## The Cauchy problem

Consider now the Cauchy problem in the whole space

$$\begin{cases} u_t - \Delta u - \frac{\lambda}{|x|^2} u = 0 & \text{in} \quad \mathbf{R}^N \times (0, \infty) \\ u(x,0) = u^0(x) \in L^2(\mathbf{R}^N) & \text{in} \quad \Omega. \end{cases}$$

When $\lambda \leq \lambda_*$ the problem is well-posed because of the Hardy inequality. The equation generates a semigroup of contractions in $L^2(\mathbf{R}^N)$.

But is there any decay rate? The classical Hardy inequality does not answer to this question because of the lack of Hardy-Poincaré version.

To overcome this difficulty we perform the similarity transformation:

$$w(y,s) = t^{N/4} u(t^{1/2} y, t); \quad s = \log(t+1).$$

The equation then reads:

$$w_s - \Delta_y w - \frac{1}{2} y \cdot \nabla w - \frac{N}{4} w - \lambda \frac{w}{|y|^2} = 0.$$

The heat equation in the similarity variables is well-behaved in the weighted spaces $L^2(K) = \{\int f^2(y)K(y)dy < \infty\}$ (see M. Escobedo & O. Kavian, 1987, ...), with $K(y) = \exp(|y|^2/4)$.
We prove the following sharp Hardy-Poincaré inequality in these weighted spaces:

$$\frac{N+2}{4}\int f^2 K dy + \frac{(N-2)^2}{4}\int \frac{f^2}{|y|^2} K dy \leq \int |\nabla f|^2 K dy$$

This yields the exponential decay rate for the evolution in similarity variables in $L^2(K)$ even for $\lambda = \lambda^*$. Returning to the original variables, for $N \geq 3$ and $\lambda = \lambda^*$ we get:

$$||u(t)||_{L^2(\mathbf{R}^N)} \leq Ct^{-1/2}||u_0||_{L^2(K)}.$$

## Control of heat processes:

Once the well-posedness of these problems is well-understood, one can adopt these techiques, and combine them with the already existing ones in Control Theory, to extend the existing control results to models with singularities. This applies to both the heat and the wave equation.

For instance, consider:

$$\begin{cases} u_t - \Delta u - \lambda \frac{u}{|x|^2} = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases}$$

$1_\omega$ denotes the characteristic function of the subset $\omega$ of $\Omega$ where the control is active.

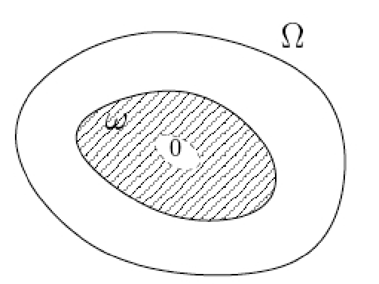
We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$:

$$\lambda < \lambda_* \Rightarrow u \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right).$$

$$\lambda = \lambda_* \Rightarrow u \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; \mathcal{H}(\Omega)\right).$$

$$u = u(x,t) = solution = state, \; f = f(x,t) = control$$

We assume that the control subdomain contains and annulus:




More recently S. Ervedoza [2] has removed this assumption obtaining the same results for general subdomains $\omega$ as in the context of the heat equation: $\lambda = 0$.

[2]S. Ervedoza, Control and Stabilization Properties for a Singular Heat Equation with an Inverse-Square Potential, Communications in Partial Differential Equations, 33: 1996Ð2019, 2008

We address the problem of null controllability: For all $u^0 \in L^2(\Omega)$ show the existence of $f \in L^2(\omega \times (0, T)$ such that:

$$u(T) \equiv 0.$$

Only makes sense if $\lambda \leq \lambda_*$.

The main result (J. Vancostenoble & E. Z., JFA, 2008; S. Ervedoza, Comm. PDE, 2008):

**Theorem**

*For all $T > 0$, annular domain $\omega$ and $\lambda \leq \lambda_*$ null controllability holds.*

Note that, due to the regularizing effect, the subtle change in the functional setting between the cases $\lambda < \lambda_*$ and $\lambda = \lambda_*$ does not affect the final control result.

The control, $f = \tilde{\varphi}$, where $\tilde{\varphi}$ minimizes:

$$J_0(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) u^0 dx$$

among the solutions of the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi - \lambda \frac{\varphi}{|x|^2} = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(0,x) = \varphi^0(x) & \text{in } \Omega. \end{cases}$$

The key ingredient, needed to prove its coercivity, is the observability inequality:

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega).$$

The main tool for obtaining such estimates are the Carleman inequalities as developed by Fursikov and Imanuvilov (1996).[3]

Goal: Combine, as done in the well-posedness of the Cauchy and boundary value problems, Hardy and Carleman inequalities.

[3] A. V. Fursikov and O. Yu. Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series # 34, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, 1996.

## Sketch of the proof:

Step 1. *Heat equation.*

Introduce a function $\eta^0 = \eta^0(x)$ such that:

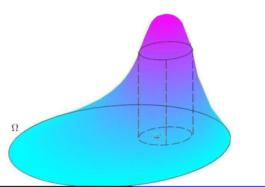
$$\begin{cases} \eta^0 \in C^2(\bar{\Omega}) & \\ \eta^0 > 0 & \text{in} \quad \Omega, \; \eta^0 = 0 \quad \text{in} \quad \partial\Omega \\ \nabla\eta^0 \neq 0 & \text{in} \quad \overline{\Omega\backslash\omega}. \end{cases} \tag{1}$$

Let $k > 0$ such that $k \geq 5 \max_{\bar{\Omega}} \eta^0 - 6 \min_{\bar{\Omega}} \eta^0$ and let

$$\beta^0 = \eta^0 + k, \bar{\beta} = \frac{5}{4} \max \beta^0, \; \rho^1(x) = e^{\lambda\bar{\beta}} - e^{\lambda\beta^0}$$

with $\lambda, \bar{\beta}$ sufficiently large. Let be finally

$$\gamma = \rho^1(x)/(t(T-t)); \rho(x,t) = \exp(\gamma(x,t)).$$



There exist positive constants $C_*, s_1 > 0$ such that

$$s^3 \int_Q \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt$$
$$\leq C_* \int_Q \rho^{-2s} \Big[ |\partial_t q - \Delta q|^2 + s^3 t^{-3}(T-t)^{-3} 1_\omega q^2 \Big] dxdt$$

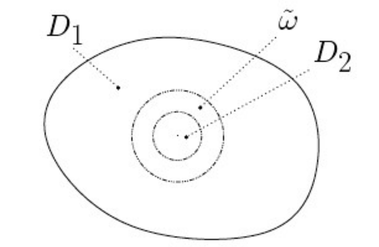
for all smooth $q$ vanishing on the lateral boundary and $s \geq s_1$.

While this inequality can be applied to deal with perturbations of the heat equations with potential in suitable $L^p$ spaces, the singularity of the quadratic potential under consideration is too large.
Thus we need to derive Carleman inequalities adapted to the presence of the singularity.

## Step 2. *Cut-off.*

Cutting-off the domain, we may:

- Apply the previous estimate in the exterior domain $|x| \geq r$ where the potential $\lambda|x|^{-2}$ is bounded;
- Concentrate in the case where $\Omega = B_1$ and $\omega$ is a neighborhood of the boundary.



Step 3. *Spherical harmonics.* To fix ideas $N = 3$, $\lambda = \lambda_* = \frac{1}{4}$.
The most singular component is the one corresponding to radially symmetric solutions:

$$-\varphi_t - \varphi_{rr} - 2\frac{\varphi_r}{r} - \frac{\varphi}{4r^2} = 0.$$

After the change of variables $\psi = r^{1/2}\varphi$,

$$-\psi_t - \psi_{rr} - \frac{\psi_r}{r} = 0.$$

This is the $2-d$ heat equation for $\psi$.
The standard Carleman inequality can be applied getting:

$$\int_0^1 \psi^2(r,0) r \, dr \leq C \int_0^T \int_a^1 \psi^2 r \, drdt$$

Going back to $\varphi$ we recover the observability inequality for $\varphi$ too, in its corresponding norm:

$$\int_0^1 \varphi^2(r,0) r^2 \, dr \leq C \int_0^T \int_a^1 \varphi^2 r^2 \, drdt.$$

Step 4. *Higher order harmonics.*

Even though for higher order harmonics the elliptic operator involved is more coercive, the potential is still singular and the existing Carleman inequalities can not be derived:

$$-\varphi_t - \varphi_{rr} - 2\frac{\varphi_r}{r} - \frac{\varphi}{4r^2} + c_j\frac{\varphi}{r^2} = 0,$$

$c_j$ being the eigenvalues of the Laplace-Beltrami operator.

This can be done by making a careful choice of the Carleman weight, exploiting the monotonicity properties of the potential. [4]

[4] Argument inspired in works by P. Cannarsa, P. Martinez, J. Vancostenoble, *Carleman estimates for a class of degenerate parabolic operators*, SIAM J. Control Optim., 2008.

## The wave equation:

Under the condition $\lambda \leq \lambda_*$:

$$\begin{cases} \varphi_{tt} - \Delta\varphi - \lambda\frac{\varphi}{|x|^2} = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(0,x) = \varphi^0(x), \varphi_t(0,x) = \varphi^1(x) & \text{in } \Omega. \end{cases}$$

The energy

$$E_\lambda(t) = \frac{1}{2}\int_\Omega \left[|\varphi_t|^2 + |\nabla\varphi|^2 - \lambda\frac{\varphi^2}{|x|^2}\right] dx,$$

is conserved, and it is coercive either in $H_0^1 \times L^2$ for $\lambda < \lambda_*$, or in $\mathcal{H} \times L^2$ for $\lambda = \lambda_*$.

Classical multipliers $(x \cdot \nabla\varphi)$ can be applied in this case too (see talks by Cavalcanti and Perla-mMenzala) :

$$TE_\lambda(0) + \int_\Omega \varphi_t \Big( x \cdot \varphi + \frac{N-1}{2}\varphi \Big) dx \Big|_0^T \leq \frac{R}{2} \int_\Sigma \Big| \frac{\partial \varphi}{\partial \nu} \Big|^2 d\Sigma.$$

Furthermore, in the absence of singularity,

$$\Big| \int_\Omega \varphi_t \Big( x \cdot \varphi + \frac{N-1}{2}\varphi \Big) dx \Big|_0^T \Big| \leq 2RE_0.$$

In the present case, in principle:

- If $\lambda < \lambda_*$ this yields the observability inequality if

$$T > \frac{2R}{\Big[1 - \lambda/\lambda_*\Big]^{1/2}} : E_\lambda(0) \leq C \int_\Sigma \Big| \frac{\partial \varphi}{\partial \nu} \Big|^2 d\Sigma.$$

- This makes the observability time to tend to $\infty$ as $\lambda \to \lambda^*$. But this does not seem to agree with our intuition that the singularity should not change the velocity of propagation.

But things are better: $N = 3$, $\lambda = \lambda_* = 1/4$.
Again using spherical harmonics decomposition the most singular component is the radial one and, after the change of variables $\psi(r,t) = r^{1/2}\varphi(r,t)$, the problem reduces to

$$\psi_{tt} - \psi_{rr} - r\psi_r = 0,$$

which is the wave equation in $2-d$ in radial coordinates.
Then observability holds and we recover:

$$E_\lambda(0) \leq \int_\Sigma \left|\frac{\partial \varphi}{\partial \nu}\right|^2 d\Sigma$$

for $T > 2R$.
In a joint paper with J. Vancostenoble (SIAM J. Math. Anal., 2009) we developed this analysis in detail for the wave and Svhrödinger equations showing that the existing observability and control results can be extended up to the critical value of $\lambda$.

## Boundary singularities

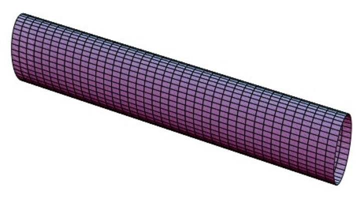
Similar problems arise when the singularity is on the boundary: $x = 0 \in \partial\Omega$. The same results apply. But there is there room for improvement of the Hardy inequalities in that case?
Consider first the thalf space $\Omega = R^N_+$. In spherical harmonics this corresponds to considering functions that oscillate in the angular variables, not radially symmetric, and therefore tinvolving only higher modes in the spherical harmocis decomposition. Actually, it is well known (Tertikas-Filippas-Tidblom, 2009) that, for all $N$:

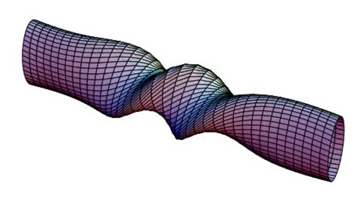
$$\frac{N^2}{4} \int_{\mathbf{R}^N_+} \frac{|u|^2}{|x|^2} dx \leq \int_{\mathbf{R}^N_+} |\nabla u|^2 dx.$$

This shows that the Hardy constant "jumps" form $(N-2)^2/4$ to $N^2/4$, when the singularity of the potential reaches the boundary. The same result holds smooth convex domains, for instance (see C. Cazacu & E. Z., where a complete analysis is developed).

## Twisted domains

Straight cylinder versus twisted one:





The cylinder $\Omega = \omega \times \mathbf{R}$ and the twisted domain $\Omega^{\theta}$, in which the cross section $\omega$ is twisted with angle $\theta$ depending on the parameter of the axis $x_3$.
In the cylinder:

$$-\Delta_D^{\Omega_\theta} - E_1 \geq 0$$

$E_1$ being the first eigenavlue of the Dirichlet Laplacian in the cross section $\omega$.

## Twisting gives Hardy inequalities and thus further decay rates.

In the twisted case the following Hardy inequality holds [5])

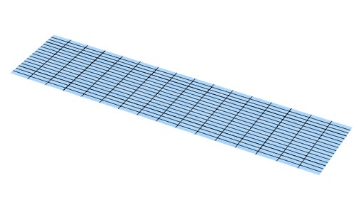
$$-\Delta_D^{\Omega_\theta} - E_1 \geq c\frac{1}{1+x_3^2}.$$

In a joint paper with D. Krejčiřik we have shown, using a careful combination of the analytical effects of twisting and similarity transformations, that the heat semigroup gains a decay rate of the order of $t^{-1/2}$ in $L^2$, because of twisting.

[5]T. Ekholm, H. Kovařik and D. Krejčiřik, A Hardy inequality in twisted waveguides, Arch. Ration. Mech. Anal. 188 (2008), 245–264.

Similar phenomena arise in other geometric contexts:

Straight strip versus twisted one:



## Open problems:

- Analyze in detail the linearization process. Back to nonlinear....
- Elliptic operators involving singular first order terms.
- Stabilization for waves and Schrödinger equations?
- Multipolar singularities (both interior and boundary ones).
- Wave equation: Better explain the propagation phenomena using bicharacteristic rays (semi-classical, Wigner, $H$-measures,...) and more geometrical tools.
- Further analyze the effect of twisting and other geometric deformations such as bending. Links with the theory of rods, shells,...?

J. L. Vázquez & E. Z.. The Hardy inequality and the asymptotic behavior of the heat equation with an inverse square potential. J. Functional Analysis, 173 (2000), 103–153.

J. Vancostenoble & E. Z. Null controllability for the heat equation with singular inverse-square potentials, J. Functional Analysis, 254 (2008), 1864–1902.

J. Vancostenoble & E. Z. Hardy inequalities, Observability and Control for the wave and Schrödinger equations with singular potentials, SIAM J. Math. Anal., Volume 41, Issue 4, pp. 1508-1532 (2009).

C. Cazacu & E. Z., Improved multipolar Hardy inequalities, Studies in Phase Space Analysis with Applications to PDEs, Progress in Nonlinear Differential Equations and Their Applications 84, M. Cicognani et al (eds.), pp. 39-57, Springer Science+Business Media New York 2013.

D. Krejčiřik & E. Z. The Hardy inequality and the heat equation in twisted tubes. J. Mathématiques Pures et Appliquées, 94 (2010)