Inverse-time design for Hamilton-Jacobi equations[1]

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[1] Based on joint work with Carlos Esteve

Let us consider the following initial-value problem:

$$\begin{cases} \partial_t u + H(D_x u) = 0, & (t,x) \in (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{HJ}$$

- The **Hamiltonian** $H : \mathbb{R}^N \to \mathbb{R}$ is usually considered to be either convex or concave (analogous results for both cases).
- The **initial datum** $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a given function.
- The **unknown** is a scalar function $u : [0,T] \times \mathbb{R}^N \to \mathbb{R}$.

Plan of the presentation:

- **Introduction to Hamilton-Jacobi equations:**
  a. A problem in calculus of variations.
  b. The Hopf-Lax formula.
  c. Viscosity solutions.
- **Inverse-time design:**
  (i) Reachability condition for the target.
  (ii) Projection on the set of reachable targets (semiconcave envelopes).
  (iii) Initial data reconstruction.
  (iv) Numerical implementation.

From a problem in calculus of variations to a Hamilton-Jacobi equation

References: L.C. Evans, *Partial Differential Equations*, Section 3.3 and 10.3.
P. Cannarsa and C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations and optimal control*, Chapter 1.

For a fixed $T > 0$, let us set the space-time domain

$$Q_T := (0, T) \times \mathbb{R}^N.$$

We are given two functions:

- The **Lagrangian**, or **running cost** $L : \mathbb{R}^N \to \mathbb{R}$.
- The **initial cost** $u_0 : \mathbb{R}^N \to \mathbb{R}$.

For any $(t, x) \in \overline{Q_T}$, we introduce the set of **admissible arcs**

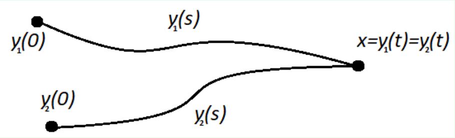
$$\mathcal{A}(t, x) := \left\{ y \in C^{0,1}([0, t]; \mathbb{R}^N) : \; y(t) = x \right\}$$

and the **cost functional**

$$J_t[y] := \int_0^t L(y'(s))ds + u_0(y(0)).$$

We consider the following **optimization problem**

$$\text{minimize } J_t[y] \text{ over all arcs } y \in \mathcal{A}(t, x).$$



The total cost depends on the velocity vector of the trajectory $y(s)$ along the interval $(0, t)$ and on the initial point $y(0)$. The terminal point is fixed to be x.

We define the **value function** $u : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ as the best final cost possible:

$$u(t,x) := \inf_{y \in \mathcal{A}(t,x)} J_t[y] = \inf_{y \in \mathcal{A}(t,x)} \left\{ \int_0^t L(y'(s))ds + u_0(y(0)) \right\}.$$

Observe that $u(0,x) = u_0(x)$.
From now on, we assume

$$\begin{cases} L \text{ is convex and } \lim_{|q|\to\infty} \dfrac{L(q)}{|q|} = +\infty \\ u_0 \in \mathrm{Lip}(\mathbb{R}^N). \end{cases} \qquad (1)$$

Using the convexity of L, we can apply Jensen's inequality to prove the following result:

Hopf-Lax formula

Under the hypotheses (1), the value function satisfies

$$u(t,x) = \min_{z \in \mathbb{R}^N} \left[u_0(z) + t\, L\left(\frac{x-z}{t}\right)\right]$$

for all $(t,x) \in (0,T] \times \mathbb{R}^N$.

Observe that, as a consequence of (1), the minimum is always attained.

Using Hopf-Lax formula we can prove Lipschitz regularity for the value function

$$|u(t',x') - u(t,x)| \leq L_0|x-x'| + L_1|t-t'|, \qquad \forall (t,x),(t',x') \in \overline{Q_T},$$

for two constants $L_0, L_1 > 0$. Hence, u is differentiable a.e. in Q_T.

The Hamilton-Jacobi equation

Let L and u_0 satisfy (1). For any $(t,x) \in Q_T$, if u is differentiable at (t,x), then it satisfies

$$u_t(t,x) + H(\nabla_x u(t,x)) = 0,$$

where

$$H(p) := \max_{q \in \mathbb{R}^N} \left[p \cdot q - L(q) \right].$$

Observe that H is the Legendre transform of L. We can write $H = L^*$, and reciprocally $L = H^*$ (recall the property of the Legendre transform $L^{**} = L$).

Remark: The value function u, given by the Hopf-Lax formula, is **not** in general the unique Lipschitz function satisfying (HJ) almost everywhere in Q_T, along with the initial condition $u(0, \cdot) = u_0$.

Definition

A uniformly continuous function $u : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is called a **viscosity solution** of (HJ) if the following two statements hold:

- u is a viscosity subsolution of (HJ): for each $\varphi \in C^\infty([0, T] \times \mathbb{R}^N)$,

$$\partial_t \varphi(t_0, x_0) + H(\nabla_x \varphi(t_0, x_0)) \leq 0$$

  whenever (t_0, x_0) is a local maximum of $u - \varphi$.
- u is a viscosity supersolution of (HJ): for each $\varphi \in C^\infty([0, T] \times \mathbb{R}^N)$,

$$\partial_t \varphi(t_0, x_0) + H(\nabla_x \varphi(t_0, x_0)) \geq 0$$

  whenever (t_0, x_0) is a local minimum of $u - \varphi$.

References: M.G. Crandall, H. Ishii, P.L. Lions, *User's guide on viscosity solutions*.
P.L. Lions, *Generalized solutions of Hamilton-Jacobi equations*.

Theorem

Let L and u_0 satisfy the hypotheses (1). The function u given by the Hopf-Lax formula is the unique viscosity solution to (HJ) satisfying $u(0,\cdot) = u_0$.

Remark: The viscosity solution can also be obtained as the limit when ε goes to 0^+ of the classical solution u_ε to the parabolic equation

$$\begin{array}{ll} \partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon + H(\nabla_x u_\varepsilon) = 0, & \text{in } Q_T, \\ u_\varepsilon(0,x) = u_0(x), & \text{in } \mathbb{R}^N. \end{array}$$

Let us define the following nonlinear operator

$$\begin{array}{rccc} S_T^+ : & \mathrm{Lip}(\mathbb{R}^N) & \longrightarrow & \mathrm{Lip}(\mathbb{R}^N) \\ & u_0 & \longmapsto & S_T^+ u_0 := u(T,\cdot) \end{array}$$

where $u(T,\cdot)$ is the unique viscosity solution to (HJ) at time $t = T$.
Using the Hopf-Lax formula we can write

$$S_T^+ u_0(x) = \min_{y \in \mathbb{R}^N} \left[u_0(y) + T\, L\left(\frac{x-y}{T} \right) \right].$$

Reference: C. Esteve and E. Zuazua, Preprint arXiv:2003.06914

Let us consider the initial-value problem:

$$\begin{cases} \partial_t u + H(D_x u) = 0, & (t,x) \in (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{HJ}$$

where u_0 is a Lipschitz function and H satisfies

$$H \in C^2(\mathbb{R}^N), \quad H_{pp}(p) > 0, \ \forall p \in \mathbb{R}^N, \quad \text{and} \quad \lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty. \tag{2}$$

Remark: Note that by the properties of the Legendre transform, $L = H^*$ is $C^2(\mathbb{R}^N)$, strictly convex and superlinear. Therefore, we can use the Hopf-Lax formula to obtain the viscosity solution to (HJ).

Goal

Given a time horizon $T > 0$ and a target function $u_T \in \text{Lip}(\mathbb{R}^N)$, construct all the initial data u_0 satisfying $S_T^+ u_0 = u_T$.

Motivation:

- In the context of the calculus of variations problem, let us suppose that we know the Lagrangian L and the value function $u(T,\cdot)$ for some time $T > 0$. Can we construct the initial cost? Is it unique?
- How do perturbations in the initial cost affect to the value function?

Can we reach any target?

For a time horizon $T > 0$ and a given target $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, let us define

$$I_T(u_T) := \left\{ u_0 \in \mathrm{Lip}(\mathbb{R}^N);\ \text{such that } S_T^+ u_0 = u_T \right\}.$$

- Our final goal is to characterize all the elements in $I_T(u_T)$.
- We start by determining whether $I_T(u_T)$ contains at least one element or not.

The natural candidate is obtained by reversing the time in the equation, considering u_T as terminal condition.

Definition

A function $w : [0, T] \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a **backward viscosity solution** to (HJ) if the function $v(t, x) := w(T - t, x)$, is a viscosity solution of

$$\partial_t v - H(D_x v) = 0, \qquad \text{in } [0, T] \times \mathbb{R}^n.$$

With this notion of solution, the terminal-value problem

$$\begin{cases} \partial_t w + H(D_x w) = 0, & (t, x) \in (0, T) \times \mathbb{R}^N \\ w(T, x) = u_T(x), & x \in \mathbb{R}^N. \end{cases} \tag{BHJ}$$

is well-posed (same arguments as for the forward problem (HJ), replacing H by $-H$).

Can we reach any target?

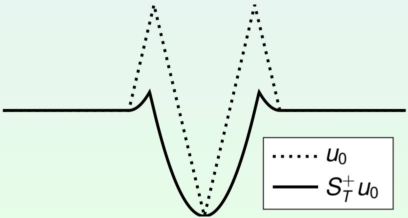
In addition, for each $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, the backward viscosity solution to (BHJ) is given by the formula

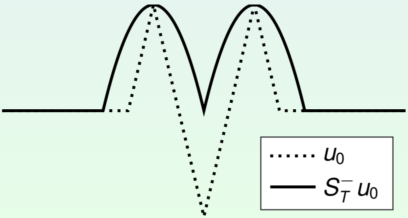
$$w(t,x) = \max_{z\in\mathbb{R}^N} \left[u_T(z) - (T-t)\, L\left(\frac{z-x}{T-t}\right)\right].$$

We can therefore define the backward operator

$$S_T^- u_T(x) = \max_{y\in\mathbb{R}^N} \left[u_T(y) - T\, L\left(\frac{y-x}{T}\right)\right],$$

which associates, to any terminal condition u_T, the viscosity solution of (BHJ) at time $t=0$.

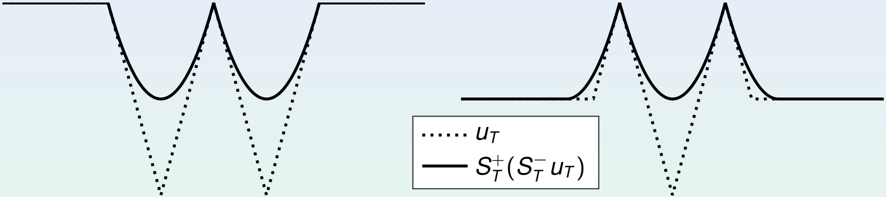




Can we reach any target?

Theorem

Let H satisfy (2), $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ and $T > 0$. Then $I_T(u_T) \neq \emptyset$ if and only if $S_T^+(S_T^- u_T) = u_T$.



Two unreachable targets

Definition

For any $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, the function

$$u_T^* := S_T^+(S_T^- u_T)$$

satisfies $I_T(u_T^*) \neq \emptyset$. We call u_T^* the **projection of u_T on the set of reachable targets**.

Geometrical properties of u_T^* (semiconcave envelopes)

Theorem

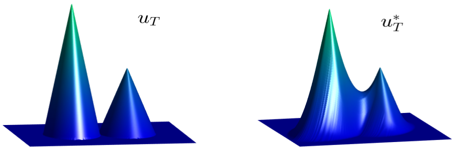
Let $N = 1$ **or** $N > 1$ and $H(p) = \langle Ap, p \rangle/2$ for some positive definite matrix A. Then, for any $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, the function $u_T^* = S_T^+(S_T^- u_T)$ is the viscosity solution to the obstacle problem

$$\min\left\{v - u_T,\ -\lambda_N\left[D^2 v - \frac{[H_{pp}(Dv)]^{-1}}{T}\right]\right\} = 0. \tag{3}$$

- Here, for a symmetric matrix X, $\lambda_N[X]$ denotes its greatest eigenvalue.
- Observe that for T large, equation (3) is an approximation of the equation for the concave envelope of u_T

$$\min\left\{v - u_T,\ -\lambda_N\left[D^2 v\right]\right\} = 0.$$

- The function u_T^* is the **smallest reachable target bounded from below by** u_T.



Here, we consider that, eventually after applying $S_T^+ \circ S_T^-$, the target u_T is reachable, i.e. $I_T(u_T) \neq \emptyset$.

Theorem

Let $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ be such that $I_T(u_T) \neq \emptyset$ and set the function $\tilde{u}_0 := S_T^- u_T$. Then, for any $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$, the two following statements are equivalent:

(i) $u_0 \in I_T(u_T)$;

(ii) $u_0(x) \geq \tilde{u}_0(x), \ \forall x \in \mathbb{R}^N$ and $u_0(x) = \tilde{u}_0(x), \ \forall x \in X_T(u_T)$,

where $X_T(u_T)$ is the subset of $\mathbb{R}^N$ given by

$$X_T(u_T) := \left\{ z - T\, H_p(\nabla u_T(z)); \ \forall z \in \mathbb{R}^N \text{ such that } u_T(\cdot) \text{ is differentiable at } z \right\}.$$

Remarks:

- If $X_T(u_T) = \mathbb{R}^N$, then $I_T(u_T) = \{\tilde{u}_0\}$. It is the case of solutions that are differentiable everywhere in $[0, T] \times \mathbb{R}^N$.
- If $X_T(u_T)$ is a proper subset of $\mathbb{R}^N$, there is no backward uniqueness. We cannot uniquely determine the initial datum.
- In any case, the initial datum is uniquely determined in $X_T(u_T)$, while in $\mathbb{R}^N \setminus X_T(u_T)$ we only have a lower bound. The information in $\mathbb{R}^N \setminus X_T(u_T)$ is partially lost at time T.

In view of the previous result, for a reachable target u_T, we need the following two ingredients in order to construct all the elements in $I_T(u_T)$:

- The function $\tilde{u}_0$ obtained as

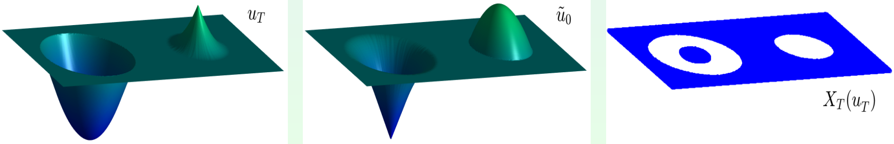
$$\tilde{u}_0(x) = S_T^- u_T(x) = \max_{y \in \mathbb{R}^N} \left[u_T(y) - T\, L\left(\frac{y - x}{T} \right) \right];$$

- and the set $X_T(u_T) \subset \mathbb{R}^N$, obtained by projecting the differentiability points of u_T by the map

$$z \longmapsto z - T\, H_p(\nabla u_T(z))$$

Once we have this two ingredients, we can construct $I_T(u_T)$ in the following way

$$I_T(u_T) = \left\{ \tilde{u}_0 + \varphi\,;\ \varphi \in \mathrm{Lip}(\mathbb{R}^n) \text{ such that } \varphi \geq 0 \text{ and } \mathrm{supp}(\varphi) \subset \mathbb{R}^n \setminus X_T(u_T) \right\}.$$



Numerical implementation

We are given a time horizon $T > 0$ and a target u_T.
Step 1: We first project u_T on the set of reachable targets by applying $S_T^+ \circ S_T^-$. Note that if the target u_T is already reachable, we will have $u_T^* = u_T$.
We can use the Hopf-Lax formula for the backward viscosity solution

$$S_T^- u_T(x) = \max_{y \in \mathbb{R}^N} \left[u_T(y) - T\,L\left(\frac{y-x}{T}\right)\right],$$

and for the forward viscosity solution

$$u_T^*(x) = S_T^+\left(S_T^- u_T(x)\right) = \min_{y \in \mathbb{R}^N} \left[S_T^- u_T(y) + T\,L\left(\frac{x-y}{T}\right)\right].$$

We can use any optimization method to approximate maximum and the minimum in the above formulae.

Using compactness estimates for the Hopf-Lax formula (see Ancona-Cannarsa-Nguyen, 2014), the maximum (resp. minimimum) in above formulae can be taken only over the ball $B(x, R_T)$, instead of all $\mathbb{R}^N$, where

$$R_T = T \sup_{|p| \leq \mathrm{Lip}[u_0]} |H_p(p)|.$$

Here $\mathrm{Lip}(u_0)$ is the Lipschitz constant of u_0.

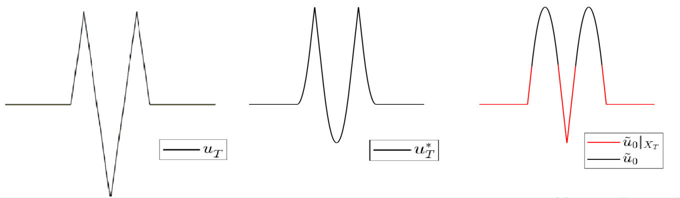
Step 2: Now, we need to compute the initial datum $\tilde{u}_0 = S_T^- u_T^*$.
However, this has already been obtained in step 1, since we have the following identity

$$S_T^- \left(S_T^+ \left(S_T^- u_T \right) \right) = S_T^- u_T, \quad \text{for all } u_T \in \mathrm{Lip}(\mathbb{R}^N).$$

(see for example [Barron et al., 1999])
Step 3: Finally, we construct the set $X_T(u_T^*)$. This is probably the most challenging part.

- One way is to identify the set of points where u_T^* is differentiable.
- There is a different (more geometrical) way to characterize $X_T(u_T^*)$ which does not use the differentiability points of u_T^*. In some situations, this can be helpful (for example if H is quadratic). See [Esteve-Zuazua, 2020] for more details.

Example:



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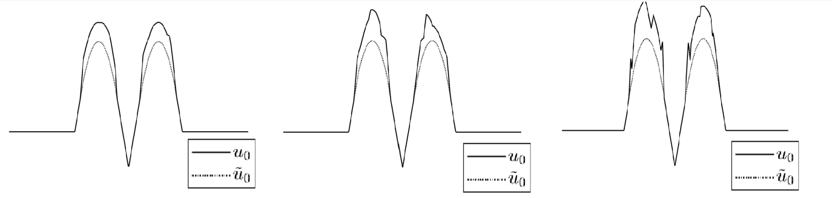

Conclusions

- The set $I_T(u_T)$ is nonempty if and only if $\tilde{u}_0 = S_T^- u_T$ satisfies $S_T^+ \tilde{u}_0 = u_T$.
- When H is quadratic or when the space-dimension is 1, $u_T^* = S_T^+(S_T^- u_T)$ is the smallest function satisfying $I_T(u_T^*) \neq \emptyset$ and $u_T^*(x) \geq u_T(x)$ for all $x \in \mathbb{R}^N$.
- If $I_T(u_T) \neq \emptyset$, then the function $\tilde{u}_0 = S_T^- u_T$ satisfies $\tilde{u}_0 \leq u_0$ for all $u_0 \in I_T(u_T)$. In addition, there exist a set $X_T(u_T) \subset \mathbb{R}^N$ where all the initial data in $I_T(u_T)$ coincide.

  Indeed, any element $u_0 \in I_T(u_T^*)$ can be written in the following way:

  $$u_0(x) = \tilde{u}_0(x) + \varphi(x),$$

  where φ is any nonnegative Lipschitz function such that $\mathrm{supp}(\varphi) \subset \mathbb{R}^N \setminus X_T(u_T^*)$.
- Backward uniqueness for (HJ) holds if and only if $X_T(u_T) = \mathbb{R}^N$.
- The solution of (HJ) at time T is invariant by increasing u_0 in $\mathbb{R}^N \setminus X_T(u_T)$.

References: - C. Esteve and E. Zuazua, *The inverse problem for Hamilton-Jacobi equations and semiconcave envelopes*, Preprint arXiv:2003.06914
- L.C. Evans, *Partial Differential Equations*, Sections 3.3 and 10.3.
- P. Cannarsa and C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations and optimal control.*
- P.L. Lions, *Generalized solutions of Hamilton-Jacobi equations.*