Controllability under constraints

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The control problem

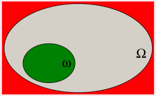
Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

$$\begin{cases} y_t - \Delta y = u 1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x, 0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

$1_\omega =$ the characteristic function of ω of Ω where the control is active, $y^0 \in L^2(\Omega)$ and $u \in L^2(Q)$ so that (5) admits an unique solution

$$y \in C\left([0, T]; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

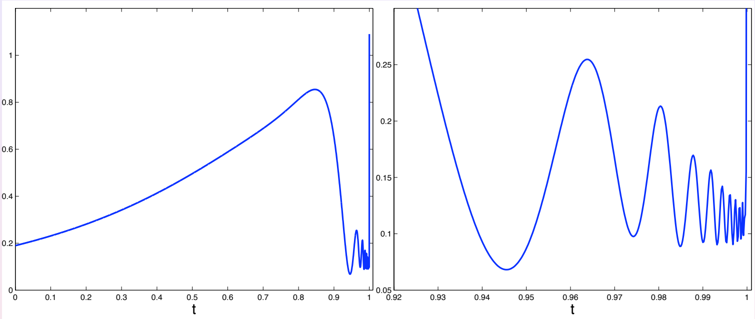
$$y = y(x, t) = \textit{solution} = \textit{state}, \; u = u(x, t) = \textit{control}$$



Goal: Drive the system to rest:

$$y(x, T) \equiv 0.$$

Numerical simulations, back to Glowinski and Lions in the 80's and 90's



This is so since controls are just restrictions to ω of solutions of the adjoint system

$$-\varphi_t - \Delta\varphi = 0$$

with initial data φ^T at time $t = T$ in very badly conditioned space, the one of the norm one observes.

Bad news

1. Controls oscillate dramatically as time approaches the final time.
2. This produces oscillations in the state too.
3. This effect is further accentuated when the time horizon T is short, as $T \to 0$.
4. This makes the controllability results of little use in many contexts in which the state represents a density (population dynamics, Math Biology).
5. This appears systematically in all models of reaction-diffusion type.

Consequences on optimal control

This not only occurs for controllability problems.

Optimal control problems that penalise the final state $||y(T)||$ will experience the same behaviour:

$$\min_{u \in L^2(\omega \times (0,T))} \frac{1}{2} \left[\int_0^T \int_\omega u^2 dxdt + K \int_\Omega |y(x,T)|^2 dx. \right]$$

The reason for this pathology is that, again, the control is, according to the Optimality System (OS), of the form

$$u = -\varphi 1_\omega$$

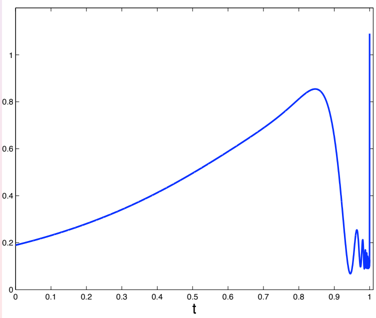
where

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma. \end{cases} \tag{2}$$

In the context of null-control, when the goal is to drive the state y to rest ($y(T) \equiv 0$) we know that the final datum φ^T of the adjoint system lies in a very large space that, in terms of the Fourier coefficients on the basis of the eigenfunctions of the Laplacian, can be written as follows:

$$\sum_{j \geq 1} |\hat{\varphi}_j^T|^2 \exp(-c\sqrt{\lambda_j}) < \infty,$$

and this explains the singular behaviour of controls as $t \sim T$:



Can we really expect null-control with nonnegative controls?

The answer is NO! When solving

$$\begin{cases} y_t - \Delta y = u 1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{3}$$

with $y^0 \geq 0$ and $u \geq 0$, then, by the comparison principle of solutions of the heat equation, we have that

$$y \geq z$$

where z is the solution of the heat equation without control. Thus

$$y(x, T) \geq z(x, T) > 0.$$

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And so?

Relax the non-negativity constraint of the control a little bit to

$$u \geq -\delta.$$

And then,

Theorem

For all $\delta > 0$ there exists a positive time-horizon $T(\delta) > 0$ such that the null-control of the equation can be achieved, the control being so that

$$u \geq -\delta.$$

The same result holds for semilinear dissipative heat equations, parabolic equations with variable smooth coefficients, etc.

Sketch of the proof

Goal: Actually prove that

$$||u||_\infty \leq \delta.$$

Two steps procedure:

1. *Phase 1:* Do nothing during time interval $[0, T-1]$ and let the solution decay.
2. *Phase 2:* Once $y(T-1)$ is small enough, control it to zero in the last time-interval $[T-1, T]$.

Two well-known key facts

1. *Dissipativity.*
2. *Controllability.* The controllability of the model, so that in time $[0, 1]$, the system is controllable and with a bound on the cost of control:

$$||u||_{L^\infty(\omega\times(0,1))} \leq C^*||y^0||_{L^2(\Omega)}.$$

Second proof: Control along a path of steady-states

[1] Allowing to link steady-states, solutions of

$$\begin{cases} -\Delta y = u 1_\omega & \text{in} \quad \Omega \\ y = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

along a path of controls $u(x, \gamma)$ depending continuously on a parameter $\gamma \in [0, 1]$.

Despacito!

[1] J.-M. Coron, E. Trélat, SIAM J. Control Optim. 43 (2004), no. 2, 549–569.

Conclusions

1. We cannot get to the target 0 with non-negative controls,
2. But, we can, if the target, the objective y^1 is a positive steady state solution satisfying

$$\begin{cases} -\Delta y^1 = u^1 1_\omega & \text{in} \quad \Omega \\ y^1 = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

with

$$u^1 \geq \nu > 0.$$

And the time T is long enough!!!

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Boundary control

The same results hold in the context of boundary control. In other words,

Theorem

Given a target y^1, steady state solution (harmonic function) with boundary control $u^1 \geq \nu > 0$, the heat equation can be driven to it in time T large enough with control $u \geq 0$.

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We have shown that controllability with constraints ca be achieved in long time. In fact, it is impossible to do it in short time!

Theorem

Whatever the initial datum y^0 and the steady state target y^1 associated to $u^1 \geq \nu$ is, the minimal control time under the positivity constraint is positive:

$$T_{\min} > 0,$$

except in the trivial case where $y^0 \equiv y^1$.

Proof of the waiting time

To fix ideas and without loss of generality we assume that $y^0 \equiv 0$.
The target $y^1 > 0$.
But we want to show that, if $u \geq 0$, it can not be reached in time T too short.
By duality, if $y(T) = y^1$,

$$\langle y^1, \varphi^T \rangle + \int_0^T \int_{\partial\Omega} u \frac{\partial \varphi}{\partial n} d\sigma(x) dt = 0,$$

where

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } (0,T) \times \Omega \\ \varphi = 0 & \text{on } (0,T) \times \partial\Omega \\ \varphi(T,x) = \varphi^T(x). & \text{in } \Omega \end{cases}$$

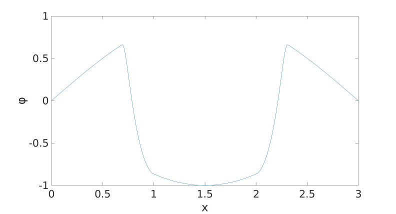
Thus, to conclude, in view of the identity

$$\langle y^1, \varphi^T \rangle + \int_0^T \int_{\partial\Omega} u \frac{\partial \varphi}{\partial n} d\sigma(x) dt = 0,$$

it suffices to find $T_0 > 0$ and a final datum $\varphi^T \in H_0^1(\Omega)$ such that, for any $T \in (0, T_0)$, the solution of the adjoint system with final datum φ^T satisfies:

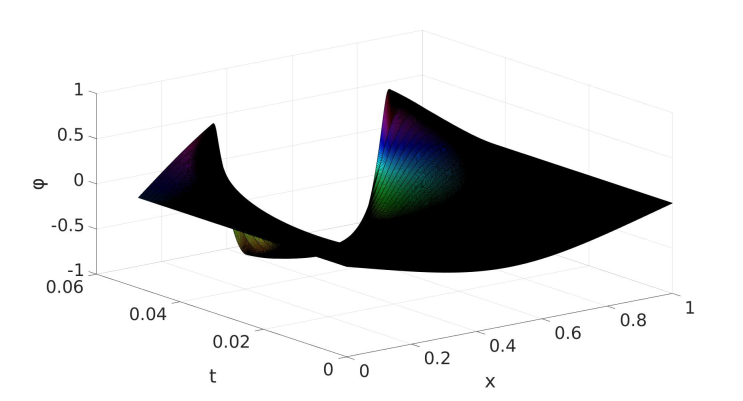
$$\begin{cases} \left(\frac{\partial \varphi}{\partial n}\right)_+ = 0 & \text{on } (0, T_0) \times \partial\Omega \\ \langle y^1, \varphi^T \rangle < 0, & \forall T \in [0, T_0). \end{cases}$$

This is assured with an initial datum of the form



Explicit estimates on the waiting time

The proof above not only yields the fact that $T_{\min} > 0$, but actually gives lower bounds on this waiting time, by a careful analysis of the behaviour of the adjoint solutions.



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Controllability in minimal time

Theorem

The system is controllable in minimal time $T_{\min}$ with a non-negative measure u as control

The proof is again a consequence of the identity:

$$\langle y^1, \varphi^T \rangle + \int_0^T \int_{\partial\Omega} u \frac{\partial \varphi}{\partial n} d\sigma(x) dt = 0,$$

now applied to the solution of the adjoint heat equation φ with Φ_1, the first eigenfunction of the Laplacian, as datum at time T:

$$\varphi = \exp(-\lambda_1(T-t))\Phi_1(x).$$

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The numerical method

1. We adopt a discrete approach: Discretise and minimise.
2. We discretise the equation both in space and time, with Δx and Δt small enough. We fix a number of time steps large enough, together with all the constraints (positivity of the boundary values, initial and final conditions).
3. We then minimise Δt using the IPOPT software.
4. We put a (very large upper bound) M on the control:

$$0 \leq u \leq M.$$

Numerical experiments

$y^0 \equiv 5$, $y^1 \equiv 1$, analytic estimate on minimal time : $\underline{T} \geq 0.165297$

See

http://cmc.deusto.es/ipopt-and-ampl-use-to-solve-time-optimal-

Numerical experiments

$y^0 \equiv 1$, $y^1 \equiv 5$, analytic estimate on minimal time : $\underline{T} \geq 0.023076$

See

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Optimality structure: And open problem

The numerical simulations above show that, apparently, the constrained controls in minimal time have the following properties:

1. They are unique.
2. They are constituted by the union of arcs where the controls vanish together with impulsional controls of Dirac delta type with support on a sequence of time instances concentrating at the final time.

This raises an interesting open problem. In view of the structure of the moment problem, can one find a solution constituted by the accumulation of a countable number of Dirac masses such that

$$u(t) = \sum_{j\geq 1} m_j \delta_{t=\tau_j}, \sum_{j\geq 1} m_j < \infty$$

with $m_j \geq 0$ and $0 < \tau_j < T$ accumulating at T as $j \to \infty$ and such that the following identity hold for all $p \geq 0$?

$$\frac{2\,y^1}{((2p+1)\pi)^2} - \frac{e^{-(2p+1)^2\pi^2 T}}{(2p+1)\pi}\, y^0_{2p+1} = 2\sum_{j\geq 1} m_j e^{-(2p+1)^2\pi^2(T-\tau_j)}.$$

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As observed by M. Tucsnak, the waiting time phenomenon for the control under nonnegativity state constraints of the linear heat equation is related to that on the control of the viscous Hamilton-Jacobi equations proved by A. Porretta and E. Z. Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 2012.

If y solves the heat equation and $y(t,x) \geq 0$, the logarithmic change of variable $z(t,x) = -\ln y(t,x)$ leads to the viscous Hamilton-Jacobi equation

$$\dot{z} - \Delta z + |\nabla z|^2 = 0 \qquad (t > 0,\ x \in \Omega), \tag{6}$$

with the initial condition $z(0,x) = -\ln yr^0(x)$ and constant target state $z^1 = -\ln yr^1$.

The waiting time for the control of this viscous Hamilton-Jacobi equation was proved using the barrier functions by Lasry and Lions. This is directly connected with the need of a minimal time for the constrained controllability of the linear heat equation. Barrier functions are achieved through the same logarithmic change of variables out of the first eigenfunction of the Laplacian.

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Conclusions

We have seen in a number of examples that:

1. There is a waiting phenomenon for controllability under constraints.
2. Constrained controllability can be achieved in a long enough time by two different methods: dissipativity and/or step-waise.
3. There is a measure control in the minimal time.
4. The numerical simulations show that the minimal time control is sparse or impulsional, composed by a sequence of diminishing Dirac deltas.
5. There is a link with the waiting time phenomena for the viscous Hamilton-Jacobi equation.

Open problems

Fully understand the sparsity structure of controls in minimal time