Observability of time-discrete conservative systems

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Outline of the talk

An abstract problem

Problem

Let $(X, \|\cdot\|_X)$ be a Hilbert space.
Consider the conservative system

$$\begin{cases} \dot{z}(t) = Az(t), \\ z(0) = z_0 \in X, \end{cases} \tag{1}$$

observed through

$$y(t) = Bz(t). \tag{2}$$

The system (A, B) is observable in time $T > 0$ if

$$k_T \|z_0\|_X^2 \leq \int_0^T \|Bz(t)\|_Y^2 \, dt \qquad \forall \; z_0 \in \mathcal{D}(A). \tag{3}$$

Can we observe the time-discrete analogues ?

Motivations:

- Time-discrete modelling and numerical analysis.
- Control Problems: Observability and Controllability are dual notions*: Control means driving solutions to rest by means of external forces acting on the system through the operator B.
- Inverse Problems: Observability is also relevant in inverse problems theory: Determine properties of the system (A for instance), through measurements that B provides.

*J. L. Lions, 1988, SIAM Review

Conservative systems

- $A : \mathcal{D}(A) \to X$ is a skew-adjoint operator.
  $\Longrightarrow$ The energy $\|z(t)\|_X$ is constant.
- A has a compact resolvent.
  $\Longrightarrow$ Its spectrum is discrete.
- Spectrum of A:

$$\sigma(A) = \{i\mu_j : \ j \in \mathbb{N}\}$$

with $(\mu_j)_{j\in\mathbb{N}}$ real numbers, associated to an orthonormal basis Ψ_j

$$A\Psi_j = i\mu_j\Psi_j$$

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Examples

- Schrödinger equation in a bounded domain: $A = -\Delta$ + BC.
- Linearized KdV equation in a bounded domain: $A = \partial_{xxx}$ + BC.
- Wave equation in a bounded domain:

$$A = \begin{pmatrix} 0 & Id \\ \Delta & 0 \end{pmatrix}.$$

- Maxwell's equation
- The Lamé system of elasticity
- ...

The observation operator

- $B : \mathcal{D}(A) \to Y$, $B \in \mathfrak{L}(\mathcal{D}(A), Y)$.

Definition

B is admissible if

$$\int_0^T \|Bz(t)\|_Y^2 \, dt \leq K_T \|z_0\|_X^2 \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{4}$$

Definition

B is observable in time $T > 0$ if

$$k_T \|z_0\|_X^2 \leq \int_0^T \|Bz(t)\|_Y^2 \, dt \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{5}$$

- When the operator B is admissible and the pair (A, B) observable $\Longrightarrow$ the norms $\left[\int_0^T \|Bz(t)\|_Y^2\, dt\right]^{1/2}$ and $\|z_0\|_X$ are equivalent.
- Often the admissibility property is a "hidden regularity" one.
  Example: The wave equation with homogeneous boundary conditions.
  When the initial data belong to $H_0^1(\Omega) \times L^2(\Omega)$, the solutions y belong to $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$. But the normal derivative $\partial y/\partial \nu$ belongs to $L^2(\partial\Omega \times (0, T))$.
  Thus, the operator $By = \partial y/\partial \nu$ is admissible from $H_0^1(\Omega) \times L^2(\Omega)$ with values in $L^2(\partial\Omega \times (0, T))$.

A natural time-discretization: *implicit midpoint scheme*

Consider the following time-discretization

$$\begin{cases} \dfrac{z^{k+1} - z^k}{\triangle t} = A\Big(\dfrac{z^{k+1} + z^k}{2}\Big), & \text{in } X, \quad k \in \mathbb{Z} \\ z^0 \text{ given}, \end{cases} \tag{6}$$

with the output function

$$y^k = Bz^k, \qquad k \in \mathbb{Z}.$$

$\rightsquigarrow$ The discrete system is conservative. Therefore, it is stable too. Being consistent, it converges as $\Delta t \to 0$ in the classical sense of numerical analysis.

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Time semi-discrete observability

Problem

To get a uniform in $\triangle t$ discrete observability estimate for the solutions of the time semi-discrete equation

$$\tilde{k}_T \left\| z^0 \right\|_X^2 \leq \triangle t \sum_{k \in (0, T/\triangle t)} \left\| B z^k \right\|_Y^2 . \tag{7}$$

The inequality (7) needs to be uniform with respect to $\triangle t$ to guarantee the convergence of discrete controls towards those of the continuous model.

The main tool : The resolvent estimate (Hautus criterion)

Theorem (Burq & Zworski, 2004, J. AMS, and Miller, 2004, JFA)

Assume A is skew-adjoint with compact resolvent, and B is admissible.
Then the following assertions are equivalent:

1. The continuous system (1)–(2) is observable in some time $T > 0$;
2. There exist constants $M, m > 0$ such that

$$M^2 \left\|(i\omega I - A)z\right\|_X^2 + m^2 \left\|Bz\right\|_Y^2 \geq \left\|z\right\|_X^2, \tag{8}$$

for all $\omega \in \mathbb{R},\ z \in \mathcal{D}(A)$.

Besides, (10) *implies observability in any time* $T > \pi M$.

Note that the estimate

$$M^2 \|(i\omega I - A)z\|_X^2 + m^2 \|Bz\|_Y^2 \geq \|z\|_X^2, \tag{9}$$

yields:

- Estimates on eigenfunctions, i. e. if $Az = i\omega z$, then

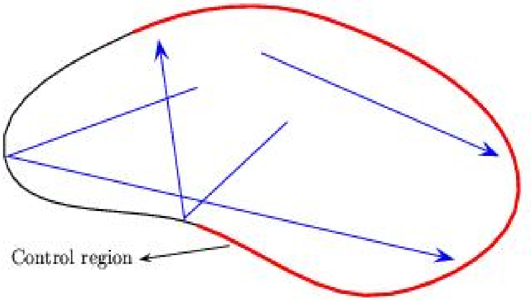
$$m^2 \|Bz\|_Y^2 \geq \|z\|_X^2 .$$

  But this does not suffice to get the observability of the time-continuous semigroup unless the spectrum fulfills an uniform gap condition (Ingham's inequality).

- Estimates on wave packets for which $\|(i\omega I - A)z\|_X \leq \delta||z||_X$ provided $M^2\delta^2 < 1$. In that case

$$m^2 \|Bz\|_Y^2 \geq (1 - M^2\delta^2)||z||_X^2.$$

So far we only got an equivalence but getting any of those estimates is not an easy matter.



Rays propagating inside the domain Ω following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case. The proof requires tools form Microlocal Analysis.

To be more precise, for the wave equation:

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ \varphi = 0 & \text{on} \quad \Sigma = \Gamma \times \times (0, T) \\ \varphi(x, 0) = \varphi^0(x), \varphi_t(x, 0) = \varphi^1(x) & \text{in} \quad \Omega. \end{cases}$$

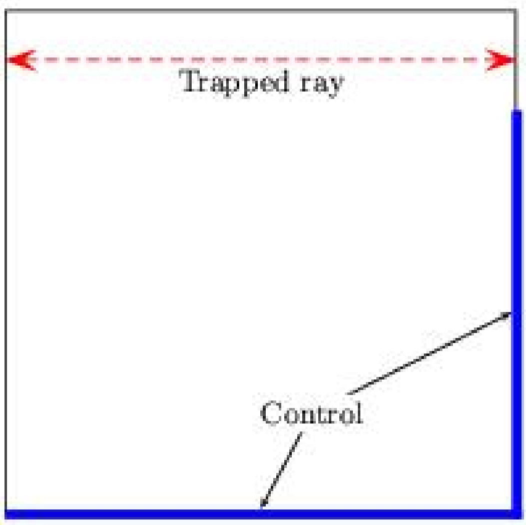
we have that

$$E_0 \leq C(\Gamma_0, T) \int_{\Gamma_0} \int_0^T \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt$$

iff the GCC is fulfilled.

A sharp discussion of this inequality requires of Microlocal analysis. Partial results may be obtained by means of multipliers ($x \cdot \nabla\varphi$, φ_t, φ,...) or Carleman inequalities.

A typical situation in which the above observability and/or resolvent estimates fail.



Resolvent estimate⇒Observability: Continuous case

Given $z^0 \in X$, set $z(t)$ the solution of the continuous system, and define, for $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$g(t) = \chi(t)z(t), \quad f(t) = g'(t) - Ag(t) = \chi'(t)z(t).$$

Then $\hat{f}(\omega) = (i\omega - A)\hat{g}(\omega)$. Apply the resolvent estimate to $\hat{g}(\omega)$:

$$\|\hat{g}(\omega)\|_X^2 \leq m^2 \left\| \widehat{Bg}(\omega) \right\|_Y^2 + M^2 \left\| \hat{f}(\omega) \right\|_X^2.$$

After integration in ω and Parseval's identity

$$\left(\int \chi(t)^2 \, dt - M^2 \int \chi'(t)^2 \, dt \right) \left\| z^0 \right\|_X^2 \leq m^2 \int \chi(t)^2 \, \|Bz(t)\|_Y^2 \, dt$$

Then choose the right χ in the right time interval....

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Observability $\Rightarrow$ Resolvent estimate: Continuous case

We need to prove

$$M^2 \left\|(i\omega I - A)z\right\|_X^2 + m^2 \left\|Bz\right\|_Y^2 \geq \left\|z\right\|_X^2, \qquad (10)$$

out of the observability inequality for the continuous problem.
Roughly, we distinguish two cases:

- When z is close to an eigenfunction with eigenvalue $i\omega$: Then $\|(i\omega I - A)z\|_X \sim 0$, $e^{At}z \sim e^{i\omega t}z$ and $\|Bz\|_Y^2 \sim \int_0^T \left\|Be^{i\omega t}z\right\|_Y^2 dt \sim \int_0^T \left\|Be^{At}z\right\|_Y^2 dt \geq ||z||_X$.
- In the opposite case $\|(i\omega I - A)z\|_X \sim ||z||_X$.

Statement of the main result.

Theorem

Assume that B is an admissible operator for A, that (A, B) satisfies the resolvent estimate (10) and that

$$\|Bz\|_Y \leq C_B \|Az\| .$$

Then, for all value of the filtering parameter $\delta > 0$, there exists a time T_δ, such that for all $T > T_\delta$, there exists $k_{T,\delta} > 0$, s.t.

$$k_{T,\delta} \left\| z^0 \right\|_X^2 \leq \triangle t \sum_{k \in (0, T/\triangle t)} \left\| Bz^k \right\|_Y^2, \qquad \forall\, z^0 \in \mathcal{C}_{\delta/\triangle t},$$

where $C_{\delta/\triangle t}$ is the class of filtered of solutions whose Fourier expansion involves only the eigenvalues $\mu \leq \delta/\triangle t$.

Remarks

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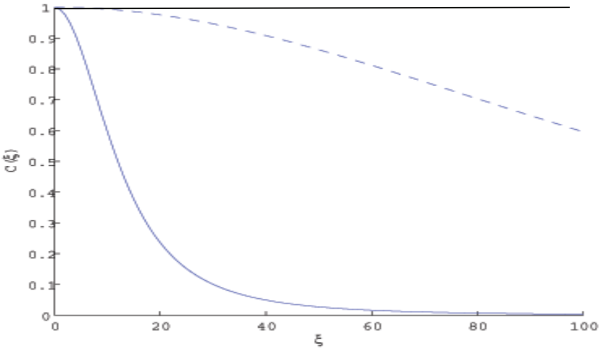
- T_δ can be chosen as

$$T_\delta = \pi \left[M^2 \left(1 + \frac{\delta^2}{4} \right)^2 + m^2 C_B^2 \frac{\delta^4}{16} \right]^{1/2},$$

which yields πM when $\delta \to 0$.

- The filtering parameter $1/\triangle t$ is at the right scale!
  - We cannot go beyond this scale as the analysis of the group veloicity shows!
  - No smallness condition of the filtering parameter δ !



Group veocity for the wave equation and its time-discrete counterpart. $C(\xi) \equiv 1$ for all values of ξ in the continuous setting. For all $\Delta t > 0$ the group velocity vanishes when $|\xi| >> 1/\Delta t$. The dispersion diagram, however, tends to the continuous one as $\Delta t \to 0$.

Consider the time-discrete wave equation. Its symbol is:

$$p_h(\tau,\xi) = -\frac{4\sin^2\frac{\tau h}{2}}{h^2} + |\xi|^2\cos(\tau h), \qquad (\tau,\xi) \in \left[-\frac{\pi}{2h}, \frac{\pi}{2h}\right] \times \mathbb{R}^d.$$

The bicharacteristic rays are defined as the solutions of the following Hamiltonian system:

$$\begin{cases} \dfrac{dx(s)}{ds} = 2\xi\cos(\tau h), & \dfrac{dt(s)}{ds} = -\dfrac{2\sin(\tau h)}{h} - |\xi|^2 h\sin(\tau h), \\ \dfrac{d\xi(s)}{ds} = 0, & \dfrac{d\tau(s)}{ds} = 0. \end{cases}$$

As in the continuous case, rays are straight lines but have different direction and the velocity of propagation.

Given $x_0 = (x_{0,1}, \cdots, x_{0,d}) \in \Omega$, $t_0 = 0$ and the initial microlocal direction $(\tau_0, \xi_0) = (\tau_0, \xi_{0,1}, \cdots, \xi_{0,d})$ a root of p_h, i.e.,

$$|\xi_0|^2 = \frac{4\sin^2\frac{\tau_0 h}{2}}{h^2\cos(\tau_0 h)}, \qquad \tau_0 \in \left(-\frac{\pi}{2h}, \frac{\pi}{2h}\right).$$

Taking, for instance, $\xi_{0,1} = 2h^{-1}\sin\frac{\tau_0 h}{2}\cos^{-1/2}(\tau_0 h)$ and $\xi_{0,2} = \cdots = \xi_{0,d} = 0$ we get

$$\frac{dx}{dt} = \frac{dx/ds}{dt/ds} = -\frac{\cos^{3/2}(\tau_0 h)}{\cos\frac{\tau_0 h}{2}}$$

and $dx_2(t)/dt = \cdots = dx_d(t)/dt = 0$. Thus, $x_j(t)$ for $j = 2, \cdots, d$ remain constant and

$$x_1(t) = x_{0,1} - t\cos^{3/2}(\tau_0 h)\cos^{-1}\frac{\tau_0 h}{2}$$

evolves with speed $-\cos^{3/2}(\tau_0 h)\cos^{-1}\frac{\tau_0 h}{2}$, which tends to 0 when $\tau_0 h \to \frac{\pi}{2}-$, or $\tau_0 h \to -\frac{\pi}{2}+$.

This allows us to show that, as $h \to 0$, there exist rays that remain trapped on a neighborhood of x_0 for time intervals of arbitrarily large length. In order to guarantee the boundary observability these rays have to be cut-off by filtering.

Sketch of the proof:

Set $\chi \in H^1(\mathbb{R})$ and $\chi^k = \chi(k\triangle t)$. Let $g^k = \chi^k z^k$, and

$$f^k = \frac{g^{k+1} - g^k}{\triangle t} - A\Big(\frac{g^{k+1} + g^k}{2}\Big). \qquad (11)$$

One can easily check that

$$\begin{aligned} f^k &= \frac{\chi^{k+1} - \chi^k}{\triangle t}\frac{z^{k+1} + z^k}{2} + \frac{\chi^{k+1} + \chi^k}{2}\frac{z^{k+1} - z^k}{\triangle t} \\ &\quad -A\Big(\frac{\chi^{k+1} + \chi^k}{2}\frac{z^{k+1} + z^k}{2} + \frac{\chi^{k+1} - \chi^k}{2}\frac{z^{k+1} - z^k}{2}\Big) \\ &= \frac{\chi^{k+1} - \chi^k}{\triangle t}\Big(\frac{z^k + z^{k+1}}{2} - \frac{(\triangle t)^2}{4}A\Big(\frac{z^{k+1} - z^k}{\triangle t}\Big)\Big) \\ &= \Big(\frac{\chi^{k+1} - \chi^k}{\triangle t}\Big)\Big(I - \frac{(\triangle t)^2}{4}A^2\Big)\Big(\frac{z^k + z^{k+1}}{2}\Big). \end{aligned} \qquad (12)$$

Compare to the following identity of the time-continuous case:

$$f = \chi' z.$$

Then

$$\left\|f^k\right\|_X^2 \leq \Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2 \left\|\frac{z^0+z^1}{2}\right\|_X^2 \Big(1+\frac{\delta^2}{4}\Big), \qquad (13)$$

provided

$$||Az|| \leq \frac{\delta}{\Delta t}||z||.$$

This requires however filtering the high frequencies. In other words, this holds within the class of solutions involving only the eigenfunctions corresponding to euigenvalues $\mu \leq \delta/\Delta t$.

A slightly more general statement

The filtering parameters and the observation time and constants are uniform for families of operators $(A_{\Delta t}, B_{\Delta t})$ fulfilling uniform admissibility and resolvent estimates.

General conservative schemes

All this can be extended to general time-discrete conservative systems. This can be done by transforming those more general discretizatio schemes into te form of the previous one.

Consider the abstract conservative time-discrete system given by

$$z^{k+1} = \mathbb{T}_{\triangle t} z^k, \qquad y^k = Bz^k, \tag{14}$$

where $\mathbb{T}_{\triangle t}$ is a linear operator such that :

1. $\exists \lambda_{j,\triangle t}, \mathbb{T}_{\triangle t}\Psi_j = \exp(i\lambda_{j,\triangle t}\triangle t)\Psi_j$.
2. There is an explicit relation between $\lambda_{j,\triangle t}$ and μ_j:

$$\lambda_{j,\triangle t} = \frac{1}{\triangle t}\, h(\mu_j \triangle t),$$

where $h : \mathbb{R} \to (-\pi, \pi)$ is an increasing smooth function satisfying

$$\lim_{\eta \to 0} \frac{h(\eta)}{\eta} = 1.$$

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Theorem

Assume that (A, B) is admissible and observable in the continuous setting, and $B \in \mathfrak{L}(\mathcal{D}(A), Y)$.
Again, discrete observability holds uniformly in $\triangle t$ for any $z^0 \in \mathcal{C}_{\delta/\triangle t}$:

$$k_{T,\delta} \left\| z^0 \right\|_X^2 \leq \triangle t \sum_{k \in (0, T/\triangle t)} \left\| B\left(\frac{z^k + z^{k+1}}{2} \right) \right\|_Y^2.$$

Besides, we have the estimate on T_δ:

$$T_\delta \leq \pi \left[M^2 \left(1 + \tan^2 \left(\frac{h(\delta)}{2} \right) \right)^2 \sup_{|\eta| \leq \delta} \left\{ \frac{\cos^4(h(\eta)/2)}{h'(\eta)^2} \right\} + m^2 C_B^2 \sup_{|\eta| \leq \delta} \left\{ \frac{2}{\eta} \tan\left(\frac{h(\eta)}{2} \right) \right\}^2 \tan^4 \left(\frac{h(\delta)}{2} \right) \right]^{1/2}.$$

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Sketch of the proof

Idea: Put the discrete system (14) into the form

$$\frac{z^{k+1} - z^k}{\triangle t} = A_{\triangle t}\left(\frac{z^{k+1} + z^k}{2}\right), \qquad \text{in } X_{\delta,\triangle t}, \qquad k \in \mathbb{Z},$$

and apply the previous theorem. To do this it suffices to define

$A_{\triangle t}$ on the basis of the eigenvectors of A but so that the corresponding eigenvalues coincide with $\lambda_{j,\Delta t}$. This can be done by applying the transformation:

$$\lambda_{j,\triangle t} = \frac{1}{\triangle t} h(\mu_j \triangle t),$$

$\longrightarrow$ Need to prove a uniform resolvent estimate for $A_{\triangle t}$. Note however that the eigenfunctions are the same! It is just a matter of rescaling the frequency parameter ω.

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Application 1: The 4th order Gauss Method

$$\begin{cases} \kappa_i = A\Big(z^k + \triangle t \sum_{j=1}^{2} \alpha_{ij}\kappa_i\Big), \qquad i = 1,2, \\ z^{k+1} = z^k + \dfrac{\triangle t}{2}(\kappa_1 + \kappa_2), \\ z^0 \in \mathcal{C}_{\delta/\triangle t} \text{ given,} \end{cases} \quad (\alpha_{ij}) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \end{pmatrix}.$$

$\Longrightarrow$ Uniform admissibility holds for $\delta < 2\sqrt{3}$: Here,

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Application 2: The Newmark method

Consider

$$\begin{cases} \ddot{u} + A_0 u = 0, \\ (u(0), \dot{u}(0)) = v_0, \end{cases} \qquad y(t) = B\dot{u}(t),$$

where A_0 is selfadjoint.

Newmark method with parameter $\beta \geq 1/4$:

$$\begin{cases} \dfrac{u^{k+1} + u^{k-1} - 2u^k}{(\triangle t)^2} + A_0\Big(\beta u^{k+1} + (1-2\beta)u^k + \beta u^{k-1}\Big) = 0, \\ \Big(\dfrac{u^0 + u^1}{2}, \dfrac{u^1 - u^0}{\triangle t}\Big) = (u_0, v_0), \quad y^{k+1/2} = B\Big(\dfrac{u^{k+1} - u^k}{\triangle t}\Big). \end{cases}$$

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Other applications

- Boundary observation of the Schrödinger equation.
- Boundary observation of the linearized KdV equation.
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Summary: Subordination principle. (transmutation, Kannai transform,...)

Observability for time continuous systems (multipliers, nonharmonic Fourier series, Carleman inequalities, microlocal analysis...)

$\Longrightarrow$

Resolvent estimate

$\Longrightarrow$

Time-discrete observability.

Key conclusion: One does not need to again the estimates at the time-discrete level. This approach can be used as a “black box" to transfer results form the continuous to the time-discrete setting.

Fully discrete schemes

Due to the explicit estimate, we can deal with fully discrete schemes.

1. First, study the space semi-discrete equations:

$$\dot{z} = A_h z, \quad y(t) = B_h z(t)$$

   and prove that admissibility and observability hold uniformly in $h > 0$[†].

2. Second, use the previous theorem to obtain uniform observability in $h, \triangle t > 0$ for the fully discrete scheme, for instance

$$\frac{z^{k+1} - z^k}{\triangle t} = A_h \Big(\frac{z^k + z^{k+1}}{2} \Big), \quad y^k = B_h z^k.$$

[†] A lot of results exists ! See E. Z., 2005, SIAM Rev.

Open problems

- Improve the time estimate that Hautus criterion gives both in the continuous and time-discrete setting.
- Weak observability (for instance in the abence of GCC for the wave equation)? Spectral characterization ?
- Spectral characterization of the observability for non conservative systems ? Note, in particular, that for the heat equation, due its very strong dissipativity properties, observability is harder to prove.
- Does the same subordination principle apply in other contexts (dispersive estimates, for instance)?
- Non-autonomous problems in the continuous setting. Time-discrete systems with variable time-step.
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