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Waves in non smooth media

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Internal stabilization of waves:

Let ω be an open subset of Ω. Consider:

$$\begin{cases} y_{tt} - \Delta y = -y_t 1_\omega & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega, \end{cases}$$

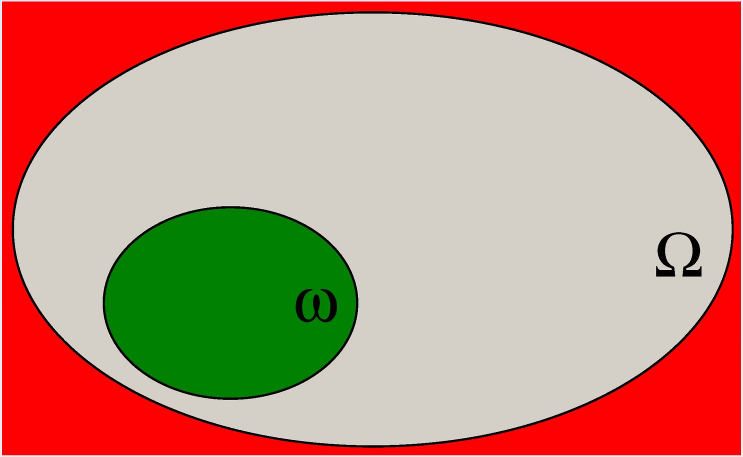
where 1_ω stands for the characteristic function of the subset ω.
The energy dissipation law is then

$$\frac{dE(t)}{dt} = -\int_\omega |y_t|^2 dx.$$

Question: Do they exist $C > 0$ and $\gamma > 0$ such that

$$E(t) \leq Ce^{-\gamma t} E(0), \quad \forall t \geq 0,$$

for all solution of the dissipative system?


Ω
ω

This is equivalent to an **observability property**[1]: There exists $C > 0$ and $T > 0$ such that

$$E(0) \leq C \int_0^T \int_\omega |y_t|^2 dxdt.$$

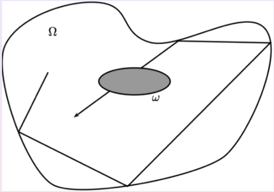
This estimate, together with the energy dissipation law, shows that

$$E(T) \leq \sigma E(0)$$

with $0 < \sigma < 1$. Accordingly the semigroup map $S(T)$ is a strict contraction. By the semigroup property one deduces immediately the exponential decay rate.

[1] J. Rauch & R. Taylor, Exponential decay of solutions to hyperbolic equations in bounded domains, Indiana Univ. Math. J . 24 (1974), 79-86.

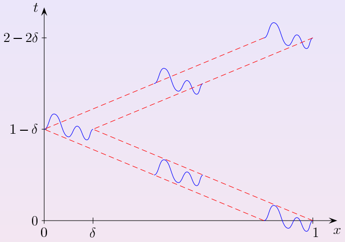
The observability inequality and, accordingly, the exponential decay property **holds if and only if the support of the dissipative mechanism, Γ_0 or ω, satisfies the so called the Geometric Control Condition (GCC)** (Ralston, Rauch-Taylor, Bardos-Lebeau-Rauch[2] ,...)

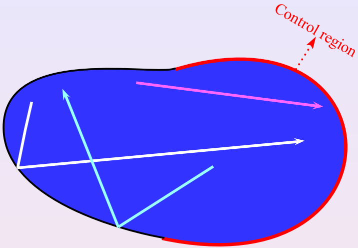


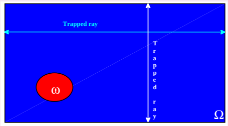
Rays propagating inside the domain Ω following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case. The proof requires tools from Microlocal Analysis.

[2] C. Bardos, G. Lebeau, and J. Rauch, "Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary", *SIAM J. Cont. Optim.*, **30** (1992), 1024–1065.

Qualitative change from $1-d$ to multi-d





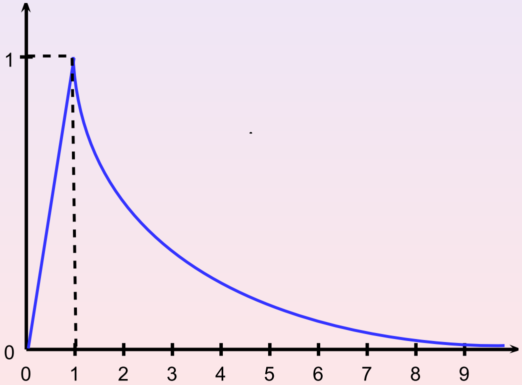


A trapped ray escaping the damping region ω makes it impossible the decay rate to be exponential. Each trajectory tends to zero as $t \to \infty$ but the decay can be arbitrarily slow.

Overdamping!

Thee decay rate γ depends on the amplitude of the damping.

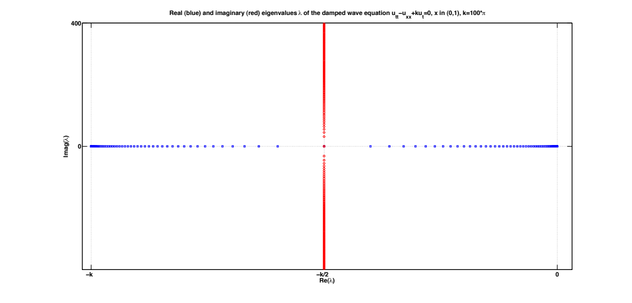
Against (?) the very first intuition, this map is not monotonic with respect to the size of the damping. A $1-d$ spectral computation for constant coefficients yields:



Some known results: $1-d$

- $1-d$: The exponential decay rate coincides with the spectral abscissa within the class of BV damping potentials. For large eigenvalues $Re(\lambda) \sim -\int_\omega a(x)dx/2$. [3] Thus:

$$\gamma_a \leq \int_\omega a(x)dx.$$



[3] S. Cox & E. Z., The rate at which the energy decays in a damped string. Comm. P.D.E. 19 (1&2). 213–243. 1994.

The singular potential

$$a(x) = \frac{2}{x}$$

produces an arbitrarily fast decay rate. [4]

Connections with:

- Transparent boundary conditions.
- Perfectly matching layers.

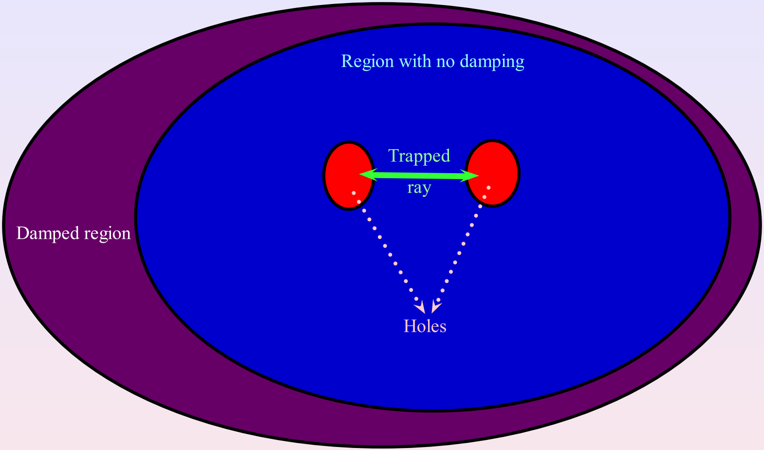
[4] C. Castro and S. Cox, Achieving arbitrarily large decay in the damped wave equation. SIAM J. Cont. Optim., 39 (6), 1748–1755, 2001.

Multi$-d$

In the multidimensional case the situation is even more complex. The decay rate is determined as the minimum of two quantities[5]:

- The spectral abscissa;
- The minimum asymptotic average (as $T \to \infty$) of the damping potential along rays of Geometric Optics.
  The later is in agreement with our intuition of waves traveling along rays of Geometric Optics.

[5] G. Lebeau, Equation des ondes amorties, Algebraic and geometric methods in mathematical physics (Kaciveli), 1993, 73-109, Math. Phys. Stud., 19, Kluwer Acad. Publ., Dordrecht, 1996.



Geometric configuration in which the spectral abscissa does not suffice to capture the decay rate. The decay rate vanishes due to a trapped ray, but the spectrum is uniformly shifted in the left complex half space.

$$\text{Truth} = \text{Spectrum} + \text{Rays}$$

$$\text{Truth} = \text{Fourier} \cup \text{D'Alembert}$$

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Consider the conservative wave equation:[6]

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ z = 0 & \text{for} \quad x \in \partial\Omega; \quad t \in (0, T) \\ z(x,0) = z^0(x), z_t(x,0) = z^1(x) & \text{in} \quad (0, \pi). \end{cases}$$

- Optimal placement problems are then of variational nature!
- It corresponds to the analysis of the behavior of the damped system with **infinitesimally small damping**.

Observability:

$$||z^0||^2_{L^2(\Omega)} + ||z^1||^2_{H^{-1}(\Omega)} \leq C(\omega, T) \int_0^T \int_\omega z^2 dxdt.$$

Inspired in previous works by, among others: P. Hébrard & A. Henrot and A. Münch, P. Pedregal & F. Periago.

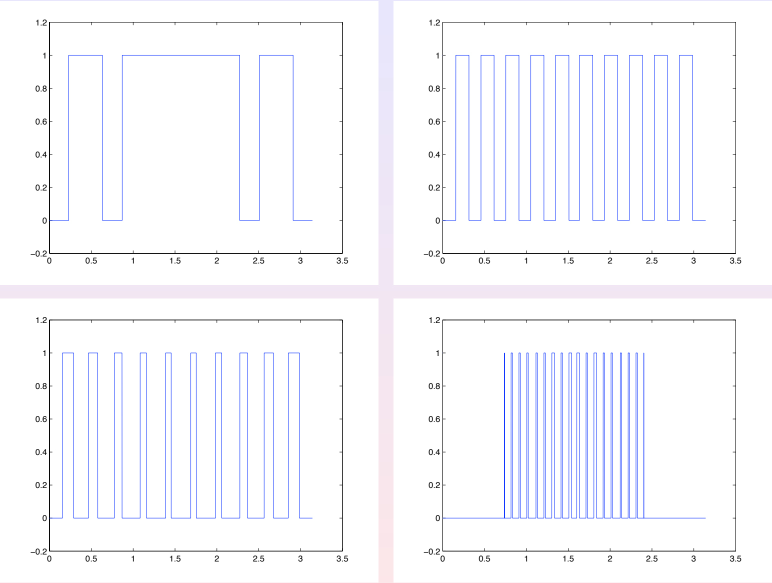
[6] Y. Privat, E. Trélat & E. Z., Optimal observation of the one-dimensional wave equation, J. Fourier Anal.Appl., **19** (2013), no. 3, 514–544.

Fourier series shows that, in general, Fourier modes are mixed in quite an complicated manner thus making the understanding of these issues complex:

$$z(t) = \sum \hat{z}_k e^{i\sqrt{\lambda_k}t}\phi_k(x).$$

Thus,

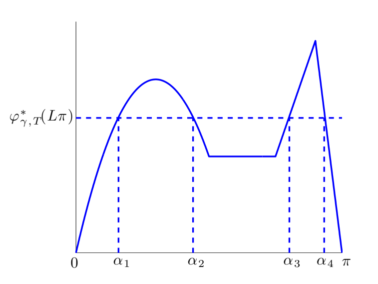
$$\int_0^T \int_\omega |z|^2 dxdt = \sum \sum \hat{z}_k \hat{z}_j \int_\omega \phi_k(x)\phi_j(x)dx \int_0^T e^{[i\sqrt{\lambda_k} - i\sqrt{\lambda_j}]t}.$$

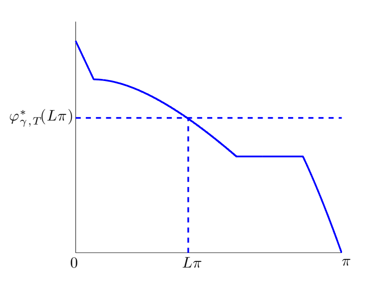


Simulations performed using AMPL + IPOPT

Main results with fixed initial data

- For initial data that are analytic (exponential decay of Fourier coefficients), there is a unique minimizer with a finite number of connected components. [7]
- The optimal set always exists but it can be a **Cantor set even for C^∞ smooth data**.





[7]Szolem Mandelbrojt, Sur un problème concernant les séries de Fourier, Boulletin de la SMF, 62 (1934), 143–150.

Reduction to a spectral problem

The problem becomes much simpler in several cases:

- The case $T = \infty$. We then look at the

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} \int_{\omega} |z|^2 dxdt.$$

- Randomizing initial data and considering the expected observability constant (Zygmund lemma, recent works by N. Burq et al.[8] [9])
- In 1-d, $\Omega = (0, \pi)$ in which case solutions are 2π-time periodic.

Cross terms vanish and we are led to the following observability problem:

$$\sum |\hat{z}_k|^2 \leq C(\omega) \sum |\hat{z}_k|^2 \int_{\omega} \phi_k^2(x) dx.$$

[8] N. Burq & N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. I. Local theory*, Invent. Math. **173** (2008), no. 3, 449–475.

[9] N. Burq & G. Lebeau, Injections de Sobolev probabilistes et applications, Annales scientifiques de l'É.N.S., Sér. 4, 46 no. 6, 2013.

These issues can be considered, as mentioned above, in two different cases:

- Fixed initial data, and therefore fixed weights $|\hat{z}_k|^2$ in ℓ^1.
- All possible initial data of finite energy. Then, the problem becomes that of finding ω so that the following minimum is maximized:

$$J(\omega) = \inf_k \int_\omega \phi_k^2(x)dx.$$

$$I = \sup_{|\omega|=L} J(\omega).$$

Warning!

This spectral criterium, is not sufficient to fully characterize the observability constant since a second microlocal one is also required. We are ignoring the rays!!!

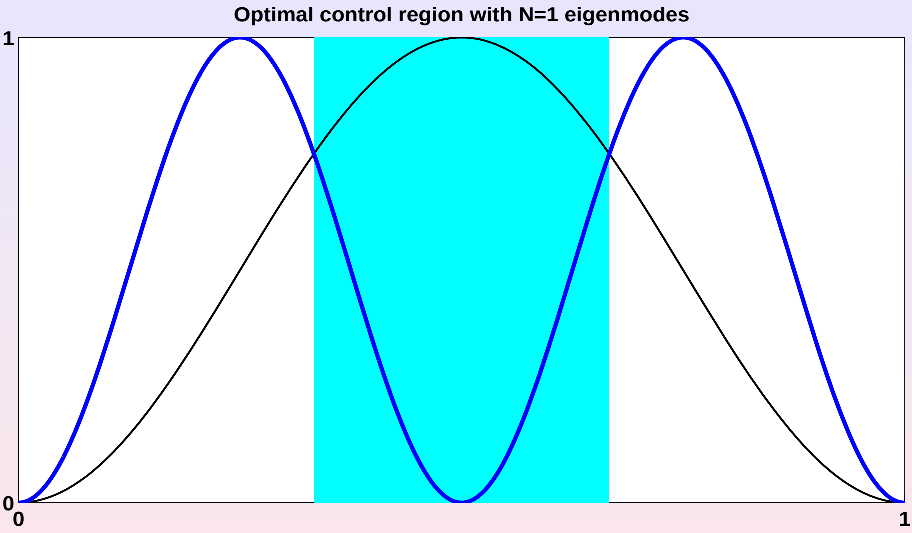
Spectral criterium = Truth /2

We are ignoring the ray contribution...

Spectral criterium: $1-d$

- Relaxation occurs [10]: the optimum is achieved by a density function $\rho(x)$ so that $\int_0^\pi \rho(x)dx = L$ and not by a measurable set with bang-bang densities (except for $L = \pi/2$). The constant density is optimal but is not the unique one.
- Spillover occurs $(1-d)$: The optimal design for the first N Fourier modes is the worst choice for the $N+1$-th one.
- No gap! The infimum over measurable sets and over densities coincides. The functional is not lower semicontinuous!!!

[10] P. Hébrard, A. Henrot, A spillover phenomenon in the optimal location of actuators, SIAM J. Control Optim. **44** (2005), 349–366.



The spillover phenomenon

- The extension of these results to the multi-dimensional case has been the object of recent work in collaboration with Y. Privat and E. Trélat.
- The full problem, without the spectral reduction, is widely open.

What about the optimal location of sensors for the complete dynamics of the wave equation?