Background on Fourier Analysis

Enrique Zuazua
FAU - AvH

Originated in a joint course delivered previously in collaboration with Aurora Maria (Polytechnique University of Bucharest)

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Wave equation and its versions

The wave equation:

$$u_{tt} - \Delta u = 0,$$

where u_t is the time derivative and Δ is the Laplace operator.

It describes small vibrations of strings $(n = 1)$, membranes $(n = 2)$ and the potential of an acoustic field $(n = 3)$. Constant coefficients, i.e. homogeneous medium.

Helmholtz equation:

$$-\Delta u = \lambda u.$$

Linear transport equation:

$$u_t + \mathbf{b} \cdot \nabla u = 0.$$

The wave equation can be written as a system of first-order equations:

$$u_t = v_x, \quad v_t = u_x.$$

Liouville equation:

$$u_t - \operatorname{div}(\mathbf{b} u) = 0.$$

Schrödinger equation of Quantum Mechanics whose solution u is a complex valued function:

$$i u_t + \Delta u = 0.$$

The beam equation:

$$u_{tt} + \Delta^2 u = 0.$$

Wave equation and its versions

Other versions of the wave equations:

- the telegraph equation
$$u_{tt} - u_{xx} + d\, u_t = 0$$
- the Airy equation
$$u_t + u_{xxx} = 0$$
- the Klein-Gordon equation
$$u_{tt} - \Delta u + u = 0.$$

The Lamé system modeling the vibrations of a $3-d$ elastic body is a coupled system of wave equations in the unknown $u = (u_1, u_2, u_3)$:

$$u_{tt} - \lambda \Delta u - (\lambda + \mu) \nabla \mathrm{div} u = 0$$

The Maxwell system describing the propagation of the electromagnetic field:

$$E_t = \mathrm{rot} B, \quad B_t = -\mathrm{rot} E, \quad \mathrm{div} B = \mathrm{div} E = 0.$$

The Eikonal equation appears in the computations of the solutions of the wave equation in Geometric Optics:

$$|\nabla u| = 1.$$

The Hamilton-Jacobi equation:

$$u_t + H(\nabla u, u, \cdot) = 0.$$

The Korteweg-de Vries (KdV) equation is a non-linear version of Airy equation allowing the analysis of the wave propagation in channels and produces solutions like the solitons:

$$u_t + uu_x + u_{xxx} = 0.$$

The Navier-Stokes equations for a homogeneous incompressible viscous flow appears in Fluid Mechanics

$$u_t - \Delta u + u \cdot \nabla u = \nabla p, \quad \text{div} u = 0$$

The Euler equations for perfect fluids, in which u is the velocity field and p is the pressure.

$$u_t + u \cdot \nabla u = \nabla p, \quad \text{div} u = 0.$$

The viscous and inviscid Burgers equation:

$$u_t + uu_x - u_{xx} = 0, \quad u_t + uu_x = 0.$$

In the last one, the solutions could develop shock waves in finite time.

Simple harmonic oscillator

Simple harmonic oscillator equation:

$$mx'' = -kx \text{ or } mx'' + kx = 0. \tag{1}$$

- $x = x(t)$ is the distance of the mass to the fixed point
- m is the mass of the particle
- k is the rigidity constant of the spring
- x', x'' are the velocity/acceleration of the particle.

Set $\omega := \sqrt{k/m}$. Then (1) can be written as

$$x''(t) + \omega^2 x(t) = 0, \quad x(0) = x^0,\ x'(0) = x^1. \tag{2}$$

SOLUTION: $x(t) = \frac{1}{2}\left(x^0 + \frac{x^1}{i\omega}\right)\exp(i\omega t) + \frac{1}{2}\left(x^0 - \frac{x^1}{i\omega}\right)\exp(-i\omega t) = x^0 \cos(\omega t) + \frac{x^1}{\omega}\sin(\omega t).$

GENERAL SOLUTION: $x(t) = \cos(\omega t + \phi)$

- A is the amplitude of the oscillation
- ω is the angular frequency; $\nu := \omega/2\pi$ is the frequency
- ϕ is the initial phase.

(2) is a second-order equation in time $\Rightarrow$ two variables in the system:

- the position of the mass, $x = x(t)$
- the velocity $v(t) := x'(t) = -\omega A \sin(\omega t + \phi)$.

The total energy $e(t) := \textit{kinetic} + \textit{potential} = \frac{1}{2}|x'(t)|^2 + \frac{\omega^2}{2}|x(t)|^2$ is conserved in time

Proof.

Multiply by x' in equation (2) $\Rightarrow 0 = (x'' + \omega^2 x)\, x' = \frac{d}{dt}\left[\frac{1}{2}|x'|^2 + \frac{\omega^2}{2}|x|^2\right] = \frac{de}{dt}(t)$. □

The trajectory $t \to (x, x')$ describes the ellipse $\omega^2 x^2 + y^2 = r^2$ in the phase plane, where $y = x'(t)$ and $r^2 = |x^1|^2 + \omega^2 |x^0|^2$.

Simple pendulum equation: $x''(t) + \omega^2 \sin(x(t)) = 0$, where $x(t)$ is the displacement angle from the equilibrium, $\omega^2 := g/l$ and l is the length of the pendulum arm.

Small oscillation assumption: If the oscillations about equilibrium are small, we expand $\sin(x(t))$ in power series and keep only the first one, $x(t)$. $\Rightarrow$ simple harmonic oscillator.

Harmonic oscillators in two dimensions

$$x''(t) + \omega_x^2 x(t) = 0, \quad y''(t) + \omega_y^2 y(t) = 0, \quad x(0) = x^0,\ x'(0) = x^1, \quad y(0) = y^0,\ y'(0) = y^1.$$

$\omega_x = \omega_y = \omega \Rightarrow$ GENERAL SOLUTION: $x(t) = A\cos(\omega t - \alpha)$ and $y(t) = B\cos(\omega t - \beta)$.

Set $\delta := \alpha - \beta$. Then $\boxed{B^2x^2 - 2AB\cos(\delta)xy + A^2y^2 = A^2B^2\sin^2(\delta)}$.

- $\delta = \pm\pi/2 \Rightarrow$ the trajectory (x, y) is the ellipse $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$.
- $\delta = 0 \Rightarrow$ the trajectory is the straight line $Bx - Ay = 0$.
- $\delta = \pm\pi \Rightarrow$ the trajectory is the straight line $Bx + Ay = 0$.

$\omega_x \neq \omega_y \Rightarrow (x, y)$ is a Lissajous curve. It is closed iff $\omega_x/\omega_y \in \mathbb{Q}$, otherwise is open and $(x, y)(\mathbb{R}^+)$ is dense in the rectangle.

Let us consider two solutions $x_1(t)$ and $x_2(t)$ of the harmonic oscillator:

$$x_1(t) := A_1 \exp(i(\omega_1 t + \phi_1)) \text{ and } x_2(t) := A_2 \exp(i(\omega_2 t + \phi_2)).$$

- When $\omega_1/\omega_2 \in \mathbb{Q} \Rightarrow$ the superposition is a periodic movement of frequency $\omega = (\omega_1, \omega_2)$.
- When $\omega_1/\omega_2 \in \mathbb{R} \setminus \mathbb{Q} \Rightarrow$ the superposition of x_1 and x_2 has no temporal periodicity.

If the two frequency are close, i.e. $\omega_2 = \omega_1 + \Delta\omega$, then

$$x(t) := x_1(t) + x_2(t) = [A_1 \exp(i\phi_1) + A_2 \exp(i(\phi_2 + \Delta\omega t))] \exp(i\omega_1 t) = A(t) \exp(i(\omega_1 t + \phi(t))),$$

$$A(t) := \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos(\phi_1 - \phi_2 - \Delta\omega t)}, \quad \tan(\phi(t)) := \frac{A_1 \sin\phi_1 + A_2 \sin(\phi_2 + \Delta\omega t)}{A_1 \cos\phi_1 + A_2 \cos(\phi_2 + \Delta\omega t)}.$$

Damped harmonic oscillator

The damped harmonic oscillator equation

$$mx'' + Rx' + kx = 0, \quad x(0) = x^0, \ x'(0) = x^1. \tag{3}$$

- $R > 0$ is the mechanical resistence constant.

The general solution of (3) is a superposition of fundamental solutions obtained by solving the characteristic equation

$$m\lambda^2 + R\lambda + k = 0. \tag{4}$$

Two roots

$$\lambda_\pm = \frac{-R \pm \sqrt{R^2 - 4mk}}{2m} \tag{5}$$

The solution

$$x(t) = \alpha_+ \exp(t\lambda_+) + \alpha_- \exp(t\lambda_-), \quad \alpha_+ = \frac{x^1 - \lambda_- x^0}{\lambda_+ - \lambda_-}, \quad \alpha_- = \frac{\lambda_+ x^0 - x^1}{\lambda_+ - \lambda_-}.$$

- UNDERDAMPING. When $0 < R < 2\sqrt{mk}$, $\lambda_\pm \in \mathbb{C}$ of real part $-R/2m$. The solutions of (3) are harmonic oscillations exponentially damped at rate $R/2m$:

$$x(t) = \exp(-Rt/2m)\left[\alpha_+ \exp(it\sqrt{4mk - R^2}/2m) + \alpha_- \exp(-it\sqrt{4mk - R^2}/2m)\right].$$

- OVERDAMPING. When $R > 2\sqrt{mk}$, $\lambda_\pm \in \mathbb{R}$ and the solution does not oscillate. The decay rate of the solution of (3) is λ_+:

$$x(t) = \exp(-Rt/2m)\left[\alpha_+ \exp(t\sqrt{R^2 - 4mk}/2m) + \alpha_- \exp(-t\sqrt{R^2 - 4mk}/2m)\right].$$

- CRITICAL DAMPING. When $R = 2\sqrt{mk}$, $\lambda_+ = \lambda_-$ and the fundamental solutions are $\exp(-Rt/2m)$ and $t\exp(-Rt/2m)$:

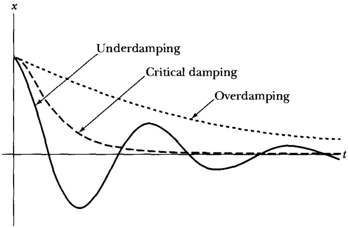
$$x(t) = (\alpha_+ + \alpha_- t)\exp(-Rt/2m) = \left(x^0 + (x^1 + Rx^0/2m)t\right)\exp(-Rt/2m).$$

The decay rate $\gamma(R)$ is as follows:

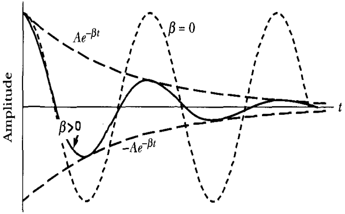
$$\gamma(R) = \begin{cases} \frac{R}{2m}, & \text{when} \quad 0 < R < 2\sqrt{mk} \\ \frac{R}{2m} - \frac{\sqrt{R^2 - 4mk}}{2m}, & \text{when} \quad R > 2\sqrt{mk}. \end{cases}$$

Properties of the decay rate:

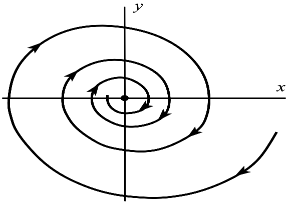
- It is increasing when $0 < R < 2\sqrt{mk}$ and decreasing when $R > 2\sqrt{mk}$
- Its maximum value is when $R = 2\sqrt{mk}$, for which $\gamma = \sqrt{\frac{k}{m}}$. However, the maximal rate is not attained since the solution involves a factor t.
- When $R \to \infty$, the decay rate tends to zero. This phenomenon that, when R is larger than the critical value $2\sqrt{mk}$, the decay rate is decreasing is called overdamping.



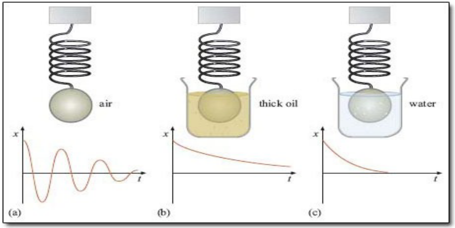
1 The three types of damped motion



2 Underdamping $X(t) = A\exp(-\beta t)\cos(\omega t)$



3 Phase portrait damped oscillator



4 (a) underdamping; (b) overdamping; (c) critical damping

Oscillators with sinusoidal driven forces

$$x''(t) + 2\beta x'(t) + \omega^2 x(t) = A\cos(\widetilde{\omega} t), \quad x(0) = x^0,\ x'(0) = x^1.$$

$$\text{SOLUTION } x(t) = \text{general} + \text{particular} := x_g(t) + x_p(t),$$

where $\quad x_g(t) = \exp(-\beta t)[C_+ \exp(t\sqrt{\beta^2 - \omega^2}) + C_- \exp(-t\sqrt{\beta^2 - \omega^2})]$

and $\quad x_p(t) = C\cos(\widetilde{\omega} t - \delta)$.

$x_p(t)$ is a solution iff, for all $t > 0$,

$$\left[A - C\left((\omega^2 - \widetilde{\omega}^2)\cos(\delta) + 2\widetilde{\omega}\beta\sin(\delta)\right)\right]\cos(\widetilde{\omega} t) = C\left((\omega^2 - \widetilde{\omega}^2)\sin(\delta) - 2\widetilde{\omega}\beta\cos(\delta)\right)\sin(\widetilde{\omega} t).$$

Thus, $\quad \tan(\delta) = \dfrac{2\widetilde{\omega}\beta}{\omega^2 - \widetilde{\omega}^2}, \quad \sin(\delta) = \dfrac{2\widetilde{\omega}\beta}{\sqrt{(\omega^2 - \widetilde{\omega}^2)^2 + 4\widetilde{\omega}^2\beta^2}}, \quad \cos(\delta) = \dfrac{\omega^2 - \widetilde{\omega}^2}{\sqrt{(\omega^2 - \widetilde{\omega}^2)^2 + 4\widetilde{\omega}^2\beta^2}}$

and $\quad C = \dfrac{A}{\sqrt{(\omega^2 - \widetilde{\omega}^2)^2 + 4\widetilde{\omega}^2\beta^2}}$.

$C_\pm$ are determined from the initial data, solving the system

$$C_+ + C_- = x^0 - C\cos(\delta) \text{ and } C_+(-\beta + \sqrt{\beta^2 - \omega^2}) + C_-(-\beta - \sqrt{\beta^2 - \omega^2}) = x^1 - C\widetilde{\omega}\sin(\delta),$$

so that $\quad C_\pm = \dfrac{1}{2}\left(x^0 - C\cos(\delta) \pm \dfrac{1}{\sqrt{\beta^2 - \omega^2}}\left(x^1 - C\widetilde{\omega}\sin(\delta) + \beta(x^0 - C\cos(\delta))\right)\right)$.

Resonance phenomenon

- For a fixed ω, as $\widetilde{\omega}$ increases from 0 to ∞, δ increases from 0 (for $\widetilde{\omega} = 0$), to $\delta = \pi/2$ (for $\widetilde{\omega} = \omega$) and to $\delta = \pi$ (for $\widetilde{\omega} = \infty$).
- $x_g(t)$ are transient effects that die out for $t >> 1/\beta$ due to the effect of the damping factor $\exp(-\beta t)$.
- $x_p(t)$ represents the steady state effects containing all the information maintained for large t.
- $\omega_d = \sqrt{\omega^2 - \beta^2}$ is the damping frequency.
- For $\widetilde{\omega} < \omega_d$, the transient response $x_g(t)$ greatly distorts the sinusoidal shape of the forcing function.
- For $\widetilde{\omega} > \omega_d \Rightarrow$ little distortion of $x_p(t)$.

- The amplitude $C = C(\widetilde{\omega})$ of $x_p(t)$ is maximized when $\widetilde{\omega} = \widetilde{\omega}_r := \sqrt{\omega^2 - 2\beta^2}$=resonant frequency.
- $\omega > \omega_d > \widetilde{\omega}_r$, for all $\omega \in \mathbb{R}$ and $\beta > 0$.

$$C(\widetilde{\omega}_r) = \frac{A}{2\beta\sqrt{\omega^2 - \beta^2}}.$$

- For $\omega^2 \geq 2\beta^2$, when the damping factor $\beta \to 0 \Rightarrow C(\widetilde{\omega}_r) \to \infty \Rightarrow x(t)$ blows-up.

If neglected, resonances could produce real disasters...

Examples:

- A crystal glass can be shattered by the right note.
- Tacoma Narrows Bridge (Washington) (July-November 1940). It received the name Galloping Gertie because of the vertical movement observed by construction workers during windy conditions and collapsed in November 1940 under high wind.
- Making a child swing to swing higher by pushing it at each swing.
- Collapse of Broughton Suspension Bridge (1826-1831), due to soldiers walking in step.
- Collapse of Königs Wusterhausen Central Tower (Germany) during the storm Quimburga in November 1972.
- Millenium Bridge, London.
- Resonant rings.
- Earthquakes. During the Mexico City earthquake in 1985, the majority of the many buildings which collapsed during were around 20 stories tall. These 20 story buildings were in resonance with the frequency of the earthquake. Other buildings, of different heights and with different vibration characteristics, were often found undamaged, even though they were located right next to the damaged 20 story buildings.

Coupled oscillators

First example. Two mass points fixed to two walls by springs with stiffness constant k and coupled by a third spring of stiffness k_c. Problem. Find the general solution for the motion of the two masses whose positions are $x_1(t)$ and $x_2(t)$.

We apply Newton's second law to obtain two coupled second-order differential equations:

$$mx_1''(t) + kx_1(t) - k_c(x_2(t) - x_1(t)) = 0, \quad mx_2''(t) + kx_2(t) - k_c(x_1(t) - x_2(t)) = 0,$$

with initial conditions $x_1(0) = x_1^0$, $x_1'(0) = x_1^1$, $x_2(0) = x_2^0$ and $x_2'(0) = x_2^1$. We rewrite these equations as

$$x_1''(t) + \frac{k + k_c}{m} x_1(t) - \frac{k_c}{m} x_2(t) = 0, \quad x_2''(t) + \frac{k + k_c}{m} x_2(t) - \frac{k_c}{m} x_1(t) = 0. \tag{6}$$

To solve this system, we change the unknowns x_1 and x_2 by the normal coordinates $q_1 = x_1 + x_2$ and $q_2 = x_1 - x_2$ solving the decoupled system

$$q_1''(t) + \omega_1^2 q_1(t) = 0, \quad q_2''(t) + \omega_2^2 q_2(t) = 0, \text{ with } \quad \omega_1 = \sqrt{\frac{k}{m}} \text{ and } \omega_2 = \sqrt{\frac{k + 2k_c}{m}},$$

$$\text{so that} \quad q_1(t) = C_1^1 \cos(\omega_1 t) + C_1^2 \sin(\omega_1 t) \text{ and } q_2(t) = C_2^1 \cos(\omega_2 t) + C_2^2 \sin(\omega_2 t),$$

$$\text{with} \quad C_1^1 = x_1^0 + x_2^0, \quad C_1^2 = \frac{x_1^1 + x_2^1}{\omega_1}, \quad C_2^1 = x_1^0 - x_2^0, \quad C_2^2 = \frac{x_1^1 - x_2^1}{\omega_2}.$$

Fourier modes

System (6) can be written as $X'' + AX = 0$, where $X = (x_1, x_2)$ and $A = \begin{pmatrix} \frac{k+k_C}{m} & -\frac{k_C}{m} \\ -\frac{k_C}{m} & \frac{k+k_C}{m} \end{pmatrix}$.

Eigensolutions (λ, φ) of A:

- $\lambda = \omega_1^2$, $\varphi = (1, 1)$
- $\lambda = \omega_2^2$, $\varphi = (1, -1)$.

Fourier modes:

- Symmetric mode. If $x_1(0) = x_2(0) = A$ and $x_1'(0) = x_2'(0) = 0$, then $x_1(t) = x_2(t) = A\cos(\omega_1 t)$.
- Anti-symmetric mode. If $x_1(0) = -x_2(0) = A$ and $x_1'(0) = x_2'(0) = 0$, then $x_1(t) = -x_2(t) = A\cos(\omega_2 t)$

Second example. Two equal masses m, one coupled to a wall and another one coupled to the first one by springs of rigidity k:

$$mx_1''(t) + kx_1(t) - k(x_2(t) - x_1(t)) = 0, \quad mx_2''(t) + k(x_2(t) - x_1(t)) = 0.$$

This system can be solved in terms of the Fourier modes:

- $\lambda = \omega_1^2 = \left(\frac{\sqrt{5}+1}{2}\sqrt{\frac{k}{m}}\right)^2$, $\varphi = (1, -(\sqrt{5}-1)/2)$.
- $\lambda = \omega_2^2 = \left(\frac{\sqrt{5}-1}{2}\sqrt{\frac{k}{m}}\right)^2$, $\varphi = (1, (\sqrt{5}+1)/2)$.

Wilberforce pendulum

Third example. The Wilberforce pendulum (spring) is a spectacular example of a system coupling translational and rotational harmonic motion.

z measures the displacement from equilibrium in the vertical direction and θ is the rotation:

$$m\ddot{z}(t) + kz(t) + \frac{1}{2}\epsilon\theta(t) = 0, \quad I\ddot{\theta}(t) + \delta\theta(t) + \frac{1}{2}\epsilon z(t) = 0.$$

- Set $\omega_z^2 := k/m$ and $\omega_\theta^2 := \delta/I$
- The two Fourier modes are:
- Translational mode. $\lambda = \omega_1^2 = \frac{1}{2}\left(\omega_z^2 + \omega_\theta^2 + \sqrt{(\omega_z^2 - \omega_\theta^2)^2 + \frac{\epsilon^2}{mI}}\right)$, $\varphi = (1, \sqrt{m/I})$
- Rotational mode. $\lambda = \omega_2^2 = \frac{1}{2}\left(\omega_z^2 + \omega_\theta^2 - \sqrt{(\omega_z^2 - \omega_\theta^2)^2 + \frac{\epsilon^2}{mI}}\right)$ $\varphi = (1, -\sqrt{m/I})$

The Cauchy problem for the $1-d$ wave equation

The Cauchy problem associated to the $1-d$ wave equation:

$$u_{tt} - u_{xx} = 0,\ x \in \mathbb{R},\ t > 0. \tag{7}$$

D'Alembert observed that the solutions of (7) can be written as a superposition of solutions of two transport equations, showing that the velocity of propagation is one:

$$u(x,t) = f(x+t) + g(x-t), \tag{8}$$

To show that any solution of (7) is of the form (8) is sufficient to observe that the d'Alembert operator $\partial_t^2 - \partial_x^2$ can be decomposed

$$\partial_t^2 - \partial_x^2 := (\partial_t - \partial_x)(\partial_t + \partial_x).$$

Introducing the auxiliary variable

$$v = (\partial_t + \partial_x)\, u, \tag{9}$$

the wave equation can be written as

$$(\partial_t - \partial_x)\, v = v_t - v_x = 0,$$

whose solution is

$$v(x,t) = h(x+t).$$

Equation (9) reduces to

$$u_t + u_x = h(x+t).$$

Observe that $w(t) := u(t + x_0, t)$ verifies the equation

$$w'(t) = h(2t + x_0),$$

whose solutions is (H being a primitive of h.)

$$w(t) = \frac{H(2t + x_0)}{2} + w(0) = \frac{H(2t + x_0)}{2} + u(x_0, 0).$$

Since $u(x,t) = w(t)$, with $x_0 = x - t$ we obtain an equivalent representation to (8):

$$u(x,t) = \frac{H(x+t)}{2} + u(x-t, 0).$$

This formula allows to compute the unique solution of the Cauchy problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{10}$$

using the d'Alembert formula

$$u(x,t) = \frac{\varphi(x+t) + \varphi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)\, dy. \tag{11}$$

The Fourier transform

The Fourier transform in $L^1(\mathbb{R}^n)$

If $f \in L^1(\mathbb{R}^n)$, its Fourier transform is defined as $\widehat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) \exp(-ix \cdot \xi)\, dx$

The inverse Fourier transform is defined as $\check{f}(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi) \exp(i\xi \cdot x)\, d\xi.$

Plancherel-Parseval identities: For all $f, g \in L^1 \cap L^2(\mathbb{R}^n)$,

$$(\widehat{f}, \widehat{g})_{L^2} = (\check{f}, \check{g})_{L^2} = (2\pi)^n (f, g)_{L^2} \text{ and } ||\widehat{f}||_{L^2} = ||\check{f}||_{L^2} = (2\pi)^{d/2} ||f||_{L^2}.$$

The Fourier transform in $L^2(\mathbb{R}^n)$

For any $f \in L^2$, there exists $f_k \in L^2 \cap L^1(\mathbb{R}^n)$ s.t. $f_k \to f$ in $L^2(\mathbb{R}^n)$. From the Parseval identity, $||\widehat{f_k} - \widehat{f_l}||_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2} ||f_k - f_l||_{L^2(\mathbb{R}^n)}$, i.e. $(\widehat{f_k})_k$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$.

Thus $\widehat{f_k} \to \widehat{f}$ in $L^2(\mathbb{R}^n)$ and $\widehat{f}$ is called the Fourier transform of f in $L^2(\mathbb{R}^n)$.

Properties of the Fourier transform in $L^2(\mathbb{R}^d)$:

- for all $f, g \in L^2(\mathbb{R}^n)$, $(f, \widehat{g})_{L^2} = (\widehat{f}, g)_{L^2}$.
- $\widehat{D^\alpha f}(\xi) = (i\xi)^\alpha \widehat{f}(\xi)$, for any index α s.t. $D^\alpha f \in L^2(\mathbb{R}^n)$.
- $\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.
- $\check{\widehat{f}} = f$.

Applications of the Fourier transform

Linear Schrödinger equation: $iu_t + \Delta u = 0,\ x \in \mathbb{R}^n, t > 0, \quad u(x,0) = \varphi(x)$

Set $\widehat{u}(\xi, t)$=the Fourier transform of $u(x,t)$. It verifies

$$i\widehat{u}_t - |\xi|^2 \widehat{u} = 0,\ \xi \in \mathbb{R}^n, t > 0, \quad \widehat{u}(\xi, 0) = \widehat{\varphi}(\xi).$$

$\widehat{u}(\xi, t)$ has the explicit expression $\widehat{u}(\xi,t) = \widehat{\varphi}(\xi)\exp(-i|\xi|^2 t)$. Then

$$u(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{\varphi}(\xi)\exp(-i|\xi|^2 t)\exp(i\xi\cdot x)\, d\xi = \frac{1}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} \varphi(y)\exp(i|x-y|^2/4t)\, dy.$$

Wave equation: $u_{tt} - \Delta u = 0,\ x \in \mathbb{R}^n, t > 0, \quad u(x,0) = u^0(x),\ u_t(x,0) = u^1(x)$

$\widehat{u}(\xi,t)$ verifies the equation $\widehat{u}_{tt} + |\xi|^2\widehat{u} = 0,\ \xi \in \mathbb{R}^n, t > 0, \quad \widehat{u}(\xi,0) = \widehat{u}^0(\xi)\ \widehat{u}_t(\xi,0) = \widehat{u}^1(\xi)$ and has the explicit expression (similar to d'Alembert formula)

$$\widehat{u}(\xi,t) = \frac{1}{2}\left(\widehat{u}^0(\xi) + \frac{\widehat{u}^1(\xi)}{i|\xi|}\right)\exp(+it|\xi|) + \frac{1}{2}\left(\widehat{u}^0(\xi) - \frac{\widehat{u}^1(\xi)}{i|\xi|}\right)\exp(-it|\xi|).$$

Telegraph equation: $u_{tt} - u_{xx} + 2du_t = 0,\ x \in \mathbb{R}, t > 0, \quad u(x,0) = u^0(x),\ u_t(x,0) = u^1(x)$

$\widehat{u}(\xi,t)$ verifies the equation $\widehat{u}_{tt} + |\xi|^2\widehat{u} + 2d\widehat{u}_t = 0,\ \xi \in \mathbb{R}, t > 0, \quad \widehat{u}(\xi,0) = \widehat{u}^0(\xi)\ \widehat{u}_t(\xi,0) = \widehat{u}^1(\xi)$ and has the explicit expression

$$\widehat{u}(\xi,t) = \sum_{\pm} \frac{1}{2}\left(\widehat{u}^0(\xi) \pm \frac{d\widehat{u}^0(\xi) + \widehat{u}^1(\xi)}{\sqrt{d^2 - |\xi|^2}}\right)\exp\left(t\left(-d \pm \sqrt{d^2 - |\xi|^2}\right)\right).$$

Wave equation in three dimensions

$$u_{tt} - \Delta u = 0,\ x \in \mathbb{R}^3, \quad u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x).$$

Spherical means. $U(x,t;r) := \oint_{\partial B(x,r)} u(y,t)\, d\sigma(y)$, $U(x,0;r) := U^0(x;r)$, $U_t(x,0;r) := U^1(x;r)$.

$\forall x \in \mathbb{R}^3$, $U(x,t;r)$ verifies the Euler-Poisson-Darboux equation: $U_{tt} - U_{rr} - \frac{2}{r} U_r = 0$, $r, t \in \mathbb{R}_+$.

Set $\widetilde{U} = rU$, which verifies the wave equation on the semiline: $\widetilde{U}_{tt} - \widetilde{U}_{rr} = 0,\ r > 0, \quad \widetilde{U}(x,t;0) = 0$.

$$\text{Then} \quad \widetilde{U}(x,t;r) = \frac{r}{2}(U^0(x;r+t) + U^0(x;r-t)) + \frac{1}{2}\int_{r-t}^{r+t} \widetilde{r} U^1(x;\widetilde{r})\, d\widetilde{r}.$$

$$\text{and} \quad u(x,t) = \lim_{r\to 0} U(x,t;r) = \lim_{r \to 0} \frac{\widetilde{U}(x,t;r)}{r}.$$

Kirchoff's formula: $$u(x,t) = \oint_{\partial B(x,t)} \Big(t u^1(y) + u^0(y) + \nabla u^0(y) \cdot (y-x) \Big)\, d\sigma(y).$$

Domain of dependence of u: the surface $\partial B(x,t)$.

Wave equation in two dimensions

$$u_{tt} - \Delta u = 0,\ x \in \mathbb{R}^2, \quad u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x).$$

Hadamard's method of descent. $\widetilde{u}(x,x') = u(x)$, $x' \in \mathbb{R}$, solves the $3-d$ wave equation $\widetilde{u}_{tt} - \Delta \widetilde{u} = 0$, $\widetilde{u}(x,x',0) = u(x,0) = u^0(x)$, $\widetilde{u}_t(x,x',0) = u_t(x,0) = u^1(x)$, which can be solved by the previous spherical means method.

Poisson's formula:

$$u(x,t) = \frac{1}{2} \oint_{B(x,t)} \frac{t^2 u^1(y) + t u^0(y) + t \nabla u^0(y) \cdot (y-x)}{(t^2 - |y-x|^2)^{1/2}}\, dy.$$

Fourier series

Given a function f on $x \in [-L, L]$, its Fourier series is

$$f(x) := \sum_{n\geq 0} a_n \cos(n\pi x/L) + \sum_{n\geq 1} b_n \sin(n\pi x/L),$$

where $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(y)\,dy, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(y)\cos(n\pi y/L)\,dy \text{ and } b_n = \frac{1}{L}\int_{-L}^{L} f(y)\sin(n\pi y/L)\,dy.$$

PROBLEMS:

- Does the series on the RHS converge?
- To f?
- At least $\int_{-L}^{L} f(y)\,dy$ should exist, so that one cannot compute Fourier series for $f(x) = 1/x^2$.

Although f might not be periodic, its Fourier series is periodic of period $2L$.

Definition

f is piecewise smooth if f and f' are piecewise continuous, i.e. f has a finite number of discontinuities on $(-L, L)$.

Theorem

For any piecewise smooth function f on $x \in (-L, L)$, its Fourier series converges:

- *to the periodic extension of f, $\widetilde{f}$, when $\widetilde{f}$ is continuous;*
- *to the average $(f(x+) + f(x-))/2$ when $\widetilde{f}$ has a jump discontinuity at x.*

Example 1. When f is the characteristic function of $(L/2, L)$, $a_0 = 1/4$, $a_n = -\sin(n\pi/2)/(n\pi)$ and $b_n = (\cos(n\pi/2) - \cos(n\pi))/(n\pi)$.

When f is an odd function on $(-L, L)$, $a_n = 0$ for all $n \geq 0 \Rightarrow$ sine Fourier series.

Example 2. $f(x) = 100$ on $x \in (0, L)$ and $f(x) = -100$ on $x \in (-L, 0) \Rightarrow b_n = 0$ for even n and $b_n = 400/(n\pi)$ for odd n.

Gibbs phenomenon appears at the discontinuity points when the Fourier series approximates a discontinuous f.

Example 3. $f(x) = x$ on $x \in (-L, L) \Rightarrow b_n = 2L(-1)^{n+1}/(n\pi)$.

Example 4. $f(x) = \cos(\pi x/L)$, $x \in (0, L)$, and $f(x) = -\cos(\pi x/L)$, $x \in (-L, 0) \Rightarrow b_n = 0$ for odd n and $b_n = 4n/(\pi(n^2 - 1))$ for even n.

When f is an even function on $(-L, L)$, $b_n = 0$ for all $n \geq 1 \Rightarrow$ cosine Fourier series.

Example 5. $f(x) = |x|$ on $(-L, L) \Rightarrow a_0 = L/2$ and $a_n = 2L((-1)^n - 1)/(n\pi)^2$.

The Fourier series of f is continuous on $[-L, L]$ iff f is continuous and $f(L) = f(-L)$.

The $1-d$ wave equation on an interval: Fourier method

Consider the $1-d$ wave equation describing the vibrations of a string of length π whose endpoints are fixed:

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < \pi, \quad t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), & 0 < x < \pi. \end{cases} \tag{12}$$

To find the solutions of (12), we firstly write the Fourier series of the initial data:

$$u^0(x) = \sum_{k=1}^{\infty} a_k \sin(kx), \quad u^1(x) = \sum_{k=1}^{\infty} b_k \sin(kx),$$

where the Fourier coefficients are as follows:

$$a_k = \frac{2}{\pi} \int_0^{\pi} u^0(x) \sin(kx)\, dx \text{ and } b_k = \frac{2}{\pi} \int_0^{\pi} u^1(x) \sin(kx)\, dx.$$

The solution of (12) is defined as:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin(kx). \tag{13}$$

The Fourier coefficients of the solution at time t are given by

$$u_k(t) = a_k \cos(kt) + \frac{b_k}{k} \sin(kt).$$

The Fourier coefficients $u_k(t)$ obey the simple harmonic oscillator equation:

$$u_k'' + k^2 u_k = 0.$$

The energy of each of these Fourier coefficients is conserved in time:

$$e_k(t) = \frac{1}{2}\left[|u_k'(t)|^2 + k^2|u_k(t)|^2\right].$$

The conservation law $E(t) = E(0)$ holds for the total energy of the solution u of (12):

$$E(t) = \frac{1}{2}\int_0^\pi \left[|u_x(x,t)|^2 + |u_t(x,t)|^2\right] dx.$$

This total energy has two parts: potential and kinetic energies.
This conservation law can be proved by

- Fourier series, by using the orthogonality of the trigonometric functions:

$$\int_0^\pi \sin(kx)\sin(jx)dx = \frac{\pi}{2}\delta_{jk}, \quad \int_0^\pi \cos(kx)\cos(jx) = \frac{\pi}{2}\delta_{jk},$$

  where δ_{jk} is the Kronecker delta, and the conservation of the energies e_k for each $k \geq 1$.
- the energy method, by multiplying (12) by u_t and integrating in $x \in (0,\pi)$.

The energy space is $H := H_0^1(0,\pi) \times L^2(0,\pi)$ of norm

$$|(f,g)|_H = \left[||f||^2_{H_0^1(0,\pi)} + ||g||^2_{L^2(0,\pi)} \right]^{1/2} := \left[\int_0^\pi \left(f_x^2 + g^2 \right) \, dx \right]^{1/2}.$$

Theorem

For any initial data $(u_0, u_1) \in H$ there exists an unique solution $(u, u_t) \in C([0,\infty); H)$ of (12). This solution belongs to the class

$$u \in C\left([0,\infty);\ H_0^1(0,\pi)\right) \cap C^1\left([0,\infty);\ L^2(0,\pi)\right)$$

and the corresponding energy $E(t)$ is conserved in time.

The fact that the initial data belong to $H_0^1(0,\pi) \times L^2(0,\pi)$ means

$$\sum_{k=1}^{\infty} [k^2 |a_k|^2 + |b_k|^2] < \infty.$$

In fact

$$E(0) = \frac{1}{2} \int_0^\pi [|u_x^0(x)|^2 + |u^1(x)|^2] \, dx = \frac{\pi}{4} \sum_{k=1}^{\infty} [k^2|a_k|^2 + |b_k|^2] < \infty.$$

The existence and uniqueness result Theorem 3 can be proved at least by three methods:

- the method of Fourier series (the one we used)
- semigroups theory
- Galerkin methods.

Fourier method for the wave equation in several space dimensions

The multi-dimensional wave equation on a bounded domain Ω

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \Omega, \quad t > 0 \\ u = 0, & x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & x \in \Omega. \end{cases} \tag{14}$$

The corresponding spectral problem:

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{en} \quad \Omega \\ \varphi = 0 & \text{en} \quad \partial\Omega. \end{cases} \tag{15}$$

The eigenvalues $\{\lambda_j\}_{j\geq 1}$ in (15) constitute an increasing sequence of positive numbers

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots \to \infty.$$

The first eigenvalue is simple and the following one are repeated according to their multiplicity. The sequence of eigenvectors $\{\varphi_j\}_{j\geq 1}$ form and orthonormal basis in $L^2(\Omega)$, i.e.

$$\int_\Omega \varphi_j \varphi_k \, dx = \delta_{jk}. \tag{16}$$

From (16) by multiplying the equation (15) corresponding to λ_k by φ_j and integrating in Ω, by the Green formula we obtain the orthogonality of the eigenfunctions in $H_0^1(\Omega)$

$$\int_\Omega \nabla\varphi_j \cdot \nabla\varphi_k dx = \lambda_j \int_\Omega \varphi_j \varphi_k dx = \lambda_j \delta_{jk} = \lambda_k \delta_{jk}.$$

We develop the initial data (u^0, u^1) in (27) as follows:

$$u^0(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x) \text{ and } u^1(x) = \sum_{k=1}^{\infty} b_k \varphi_k(x). \tag{17}$$

and look for the solution u of (27) in the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\varphi_k(x).$$

Remark that the coefficients $\{u_k\}$ solve the harmonic oscillator equation:

$$u_k''(t) + \lambda_k u_k(t) = 0,\ \ t > 0, \quad u_k(0) = a_k, \quad u_k'(0) = b_k,$$

and then

$$u_k(t) = a_k \cos\left(\sqrt{\lambda_k} t\right) + \frac{b_k}{\sqrt{\lambda_k}} \sin\left(\sqrt{\lambda_k} t\right).$$

The solution u of (27) admits the Fourier representation

$$u(x,t) = \sum_{k=1}^{\infty} \left(a_k \cos\left(\sqrt{\lambda_k} t\right) + \frac{b_k}{\sqrt{\lambda_k}} \sin\left(\sqrt{\lambda_k} t\right) \right) \varphi_k(x).$$

In the $1-d$ case, the eigenvalues and eigenvectors of the Laplacian on $\Omega = (0,\pi)$ are

$$\lambda_k = k^2,\ k \geq 1;\ \varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx), \quad k \geq 1.$$

Rectangular and circular membranes

- When Ω is the rectangle $\Omega := [0, L] \times [0, H]$, $\lambda_{m,n} := (m\pi/L)^2 + (n\pi/H)^2$ and $\varphi_{m,n}(x, y) := 2\sin(m\pi x/L)\sin(n\pi y/H)$.
- When Ω is the disc of radius a, λ and $\varphi(x, y) = \varphi(r, \theta) = f(r)g(\theta)$ are related to the Bessel functions.
- g satisfies the problem

$$g''(\theta) + \mu g(\theta) = 0 \text{ and periodic boundary conditions } g(\pi) = g(-\pi), \quad g'(\pi) = g'(-\pi),$$

so that $\mu = m^2$, $m \in \mathbb{N}$, $g(\theta) = \sin(m\theta)$ or $g(\theta) = \cos(m\theta)$.
- f satisfies the problem

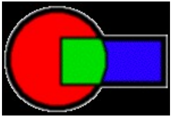
$$r^2 f''(r) + rf'(r) + (\lambda r^2 - m^2)f(r) = 0, \quad f(a) = 0, \quad |f(0)| < \infty.$$

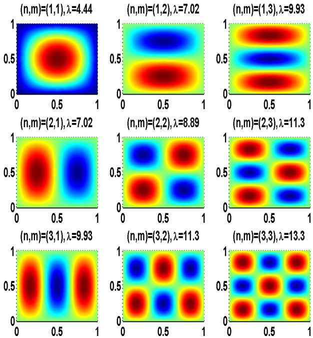
- $f(r) = \tilde{f}(z)$, with $z = \sqrt{\lambda}r \Rightarrow z^2\tilde{f}''(z) + z\tilde{f}'(z) + (z^2 - m^2)\tilde{f}(z) = 0$. Then $\tilde{f}(z) = c_1 J_m(z) + c_2 Y_m(z)$, where J_m and Y_m are Bessel functions of first/second kind of order m:

$$J_m(z) = \sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha}{\alpha!\Gamma(\alpha + m + 1)}(z/2)^{2\alpha+m} = \frac{1}{\pi}\int_0^\pi \cos(my - z\sin(y))\,dy$$

$$\text{and} \quad Y_m(z) = \frac{1}{\pi}\int_0^\pi \sin(z\sin(y) - my)\,dy.$$

- The eigenvalue λ is a solution of $J_m(\sqrt{\lambda}a) = 0$. z_{mn}:=the n-th zero of $J_m \Rightarrow \lambda_{mn} = (z_{mn}/a)^2$.
- More complex domains $\Rightarrow$ domain decomposition methods.

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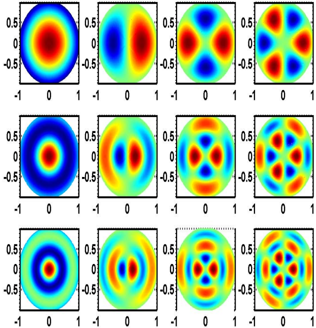
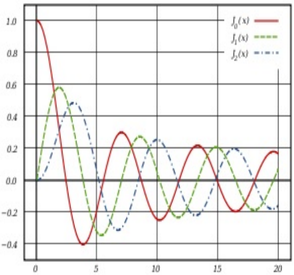
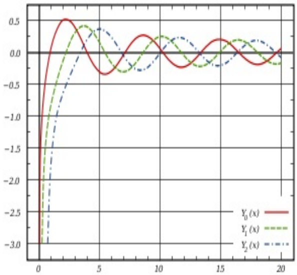
(n,m)=(1,2), λ=7.02
(n,m)=(1,3), λ=9.93
(n,m)=(2,1), λ=7.02
(n,m)=(2,2), λ=8.89
(n,m)=(2,3), λ=11.3
(n,m)=(3,1), λ=9.93
(n,m)=(3,2), λ=11.3
(n,m)=(3,3), λ=13.3




$J_0(x)$
$J_1(x)$
$J_2(x)$


$Y_0(x)$
$Y_1(x)$
$Y_2(x)$

Tools to highlight the vibrational modes

- Cymatics (from Greek word for wave) is the study of visible sound and vibration, a subset of modal phenomena. Typically the surface of a plate, diaphragm or membrane is vibrated and regions of maximum and minimum displacement are made visible in a thin coating of particles, paste, or liquid. Different patterns emerge in the medium depending on the geometry of the plate and the driving frequency.
- Chladni patterns. One of Chladni's (1756-1827) best-known achievements was to invent a technique to show the various modes of vibration of a rigid surface. A plate or membrane vibrating at resonance is divided into regions vibrating in opposite directions, bounded by lines of zero vibration called nodal lines.
- Rubens tube. Invented by Heinrich Rubens (1865-1922). A pipe is perforated periodically and sealed at both ends. One seal is attached to a frequency generator, the other to a supply of a flammable gas. The pipe is filled with the gas, and the gas leaking from the perforations is lit. The flame height is proportional to the gas flow. Based on Bernoulli's principle, the gas flow is proportional to the square root of the pressure difference between the inside and outside of the tube. Where there is oscillating pressure due to the sound waves, less gas will escape from the perforations in the tube, and the flames will be lower at those points. At the pressure nodes, the flames are higher.
- Wave pendulum.
- Interferometry.

The total energy of the solutions of the simple wave equation is also conserved in time:

$$E(t) = \frac{1}{2} \int_\Omega \left[|\nabla u(x,t)|^2 + |u_t(x,t)|^2 \right] dx$$

The energy space is: $H = H_0^1(\Omega) \times L^2(\Omega)$.

Theorem

For any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists an unique solution $(u, u_t) \in C([0,\infty); H)$ of (27), i.e.

$$u \in C\left([0,\infty); H_0^1(\Omega)\right) \cap C^1\left([0,\infty); L^2(\Omega)\right)$$

whose energy $E(t)$ is conserved in time.

The regularity of u in Theorem 4 and the fact that the Laplace operator with homogeneous Dirichlet boundary conditions is an isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ allows to conclude that $u \in C^2\left([0,\infty); H^{-1}(\Omega)\right)$. In this way, equation (27) makes sense in $H^{-1}(\Omega)$ for any $t > 0$.

Extensions and limitations of the Fourier method

The Fourier method can be apply to other situations:

- Neumann or mixed boundary conditions
- more general wave equations involving variable coefficients:

$$\rho(x)u_{tt} - \mathrm{div}(a(x)\nabla u) + q(x)u = 0,$$

  where ρ, a and q are measurable and bounded functions and ρ and a are uniformly positive, i.e. there exist $\rho_0,\ a_0 > 0$ such that

$$\rho(x) \geq \rho_0,\ a(x) \geq a_0,\ a.e.\ x \in \Omega.$$

Limitations of the Fourier method:

- nonlinear equations
- equations with coefficients depending on both x and t.

Fourier series as numerical method

The solution of the wave equation (27) can be expressed as

$$u(x,t)=\sum_{k=1}^{\infty}\left[a_k\cos\left(\sqrt{\lambda_k}t\right)+\frac{b_k}{\sqrt{\lambda_k}}\sin\left(\sqrt{\lambda_k}t\right)\right]\varphi_k(x), \tag{18}$$

where $\{\varphi_k\}_{k\geqslant 1}$ and $\{\lambda_k\}_{k\geqslant 1}$ are the eigenvalues and eigenfunctions of the Laplacian. The energy below is conserved along the trajectories:

$$E(t)=\frac{1}{2}\int_{\Omega}\left[|\nabla u(x,t)|^2+|u_t(x,t)|^2\right]\,dx,$$

so that the energy of the initial data is

$$E(0)=\frac{1}{2}\sum_{k=1}^{\infty}\left[\lambda_k|a_k|^2+|b_k|^2\right]. \tag{19}$$

The hypothesis that the initial data $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ is of finite energy is equivalent to the fact that the sequences $\left\{a_k\sqrt{\lambda_k}\right\}$ and $\left\{b_k\right\}$ belong to ℓ^2.

It seems natural to construct a numerical approximation of the solution of the wave equation in the form

$$u_N(x,t)=\sum_{k=1}^{N}\left[a_k\cos(\sqrt{\lambda_k}t)+\frac{b_k}{\sqrt{\lambda_k}}\sin(\sqrt{\lambda_k}t)\right]\varphi_k(x).$$

To prove that u_N is an approximation of the solution u of (27), we consider the difference

$$\epsilon_N := u - u_N = \sum_{k \geqslant N+1} \left[a_k \cos(\sqrt{\lambda_k} t) + \frac{b_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \right] \varphi_k(x).$$

Due to the orthogonality of the eigenvectors,

$$||\nabla \epsilon_N(t)||^2_{L^2(\Omega)} = \sum_{k \geqslant N+1} \lambda_k \left[a_k \cos(\sqrt{\lambda_k} t) + \frac{b_k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \right]^2 \leqslant \sum_{k \geqslant N+1} \left[\lambda_k |a_k|^2 + |b_k|^2 \right].$$

Due to the boundedness of the series representing $E(0)$ in (19), we conclude

$$u_N(t) \to u(t) \text{ in } C\left([0,\infty); H_0^1(\Omega)\right) \text{ as } N \to \infty. \tag{20}$$

By the same arguments,

$$u_{N,t} \to u_t(t) \text{ in } C\left([0,\infty); L^2(\Omega)\right) \text{ as } N \to \infty. \tag{21}$$

From (20)-(21) $\Rightarrow$ convergence in the energy space $H := H_0^1(\Omega) \times L^2(\Omega)$ uniformly in $t \geqslant 0$.

CONVERGENCE RATES?!?

- The previous argument does not provide any information in this sense, since the convergence of the series $E(0)$ in (19) does not allow to determine the order of convergence of the truncations.
- Take more regular initial data in (27), $(u^0, u^1) \in \left[H^2 \cap H_0^1(\Omega) \right] \times H_0^1(\Omega)$.

When $(u^0, u^1) \in \left[H^2 \cap H_0^1(\Omega)\right] \times H_0^1(\Omega)$, we have

$$||(u^0, u^1)||_{H^2 \cap H_0^1 \times H_0^1} := \sum_{k \geqslant 1} \left[\lambda_k^2 |a_k|^2 + \lambda_k |b_k|^2\right] < \infty.$$

On the other hand, we know the norm equivalence:

$$||\Delta \varphi||_{L^2(\Omega)} \approx ||\varphi||_{H^2(\Omega)}, \text{ for any } \varphi \in H^2 \cap H_0^1(\Omega),$$

which holds due to classical elliptic regularity results in the Dirichlet problem for the Laplace operator: if the domain Ω is of class C^2 and $f \in L^2(\Omega)$, then the solution u the following problem belongs to $H^2 \cap H_0^1(\Omega)$:

$$-\Delta u = f, \;\; x \in \Omega, \quad u = 0, \;\; x \in \partial\Omega.$$

On the other hand,

$$\int_\Omega \Delta \varphi_k \Delta \varphi_j \, dx = \lambda_k^2 \delta_{j,k},$$

so that

$$||\Delta u^0||_{L^2(\Omega)}^2 = \sum_{k \geqslant 1} \lambda_k^2 |a_k|^2.$$

The additional regularity of the initial data $(u^0, u^1) \in H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)$ and the increasing character of $\{\lambda_k\}$ imply:

$$\begin{aligned} ||\nabla \epsilon_N(t)||_{L^2}^2, ||\partial_t \epsilon_N(t)||_{L^2}^2 &\leqslant 2 \sum_{k \geqslant N+1} \left[\lambda_k |a_k|^2 + |b_k|^2\right] \leqslant 2 \sum_{k \geqslant N+1} \frac{1}{\lambda_k} \left[\lambda_k^2 |a_k|^2 + \lambda_k |b_k|^2\right] \\ &\leqslant \frac{2}{\lambda_{N+1}} \sum_{k \geqslant N+1} \left[\lambda_k^2 |a_k|^2 + \lambda_k |b_k|^2\right] \leqslant \frac{C}{\lambda_{N+1}} ||(u^0, u^1)||_{H^2 \cap H_0^1 \times H_0^1}^2. \end{aligned}$$

Conclusion:

$$||u-u_N||_{L^\infty(0,\infty;H_0^1(\Omega))\cap W^{1,\infty}(0,\infty;L^2(\Omega))} \leq \frac{C}{\sqrt{\lambda_{N+1}}}||(u^0,u^1)||_{H^2\cap H_0^1(\Omega)\times H_0^1(\Omega)}.$$

Weyl Theorem on the asymptotic distribution of the eigenvalues of the Laplace operator:

$$\lambda_N \sim c(\Omega)N^{2/n} \text{ as } N\to\infty.$$

Theorem

For any $(u^0,u^1)\in H^2\cap H_0^1\times H_0^1(\Omega)$ initial data in (27), u_N converges to u in the energy space $H:=H_0^1\times L^2(\Omega)$ uniformly in $t\geq 0$ at order $O\left(N^{-1/n}\right)$.

The regularity hypothesis $(u^0,u^1)\in H^2\cap H_0^1\times H_0^1(\Omega)$ is not the only possible one!

The Fourier approximation method is useful in the following cases:

- $1-d$, when one compute explicitly the spectrum of the Laplacian.
- to compute the Fourier coefficients of the initial data, one can apply quadrature formulas.
- several space dimensions when Ω is the square or the circle.

The Fourier approximation method is difficult to be applied in the following cases:

- several space dimensions on complex domains Ω (union of square+circle$\Rightarrow$domain decomposition methods)
- nonlinear equations or variable coefficients depending on both (x,t)

Other approximation methods which have not these limitations:

- finite differences, finite elements, finite volumes.

Damped wave equation

Initial boundary value problem for the damped wave equation in the bounded domain $\Omega \subset \mathbb{R}^n$, with $a > 0$:

$$\begin{cases} u_{tt} - \Delta u + a u_t = 0 & \text{in} \quad \Omega \times (0, \infty) \\ u = 0 & \text{in} \quad \partial\Omega \times (0, \infty) \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{22}$$

Writing this equation like $u_{tt} - \Delta u = -a u_t$, we see that $-a u_t$ is a force acting on the whole Ω at any time $t > 0$.

To determine the solution of (22) using the Fourier method, we firstly develop the initial data in Fourier series:

$$u^0(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x) \text{ and } u^1(x) = \sum_{k=1}^{\infty} b_k \varphi_k(x).$$

We look for a solution u of (22) as follows:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \varphi_k(x), \tag{23}$$

where $u_k(t)$ is solution for the damped harmonic oscillator:

$$u_k''(t) + \lambda_k u_k(t) + a u_k'(t) = 0,\ t > 0, \quad u_k(0) = a_k, \quad u_k'(0) = b_k. \tag{24}$$

The characteristic polynomial associated to (24) is $\mu^2 + \lambda_k + a\mu = 0$ and its roots are

$$\mu_k^{\pm} = \frac{-a \pm \sqrt{a^2 - 4\lambda_k}}{2}.$$

The solution of (24) is given by

$$u_k(t) = \alpha_+ \exp(\mu_k^+ t) + \alpha_- \exp(\mu_k^- t).$$

The constants $\alpha_\pm$ are so that

$$\alpha_+ + \alpha_- = a_k \text{ and } \mu_k^+ \alpha_+ - \mu_k^- \alpha_- = b_k.$$

When $a^2 = 4\lambda_k$ the solution of (24) is

$$u_k(t) = \alpha \exp(-at/2) + \beta t \exp(-at/2),$$

with

$$\alpha = a_k \text{ and } -\frac{a}{2}\alpha + \beta = b_k.$$

The energy of the solution of (22),

$$E(t) = \frac{1}{2} \int_\Omega \left[|\nabla u(x,t)|^2 + |u_t(x,t)|^2 \right] dx,$$

is decreasing in time. By multiplying by u_t in (22), we obtain the dissipation law of the energy:

$$\frac{dE}{dt}(t) = -a \int_\Omega |u_t(x,t)|^2 \, dx.$$

Decay rate of the energy by the Fourier method

From the orthogonality properties of the eigenvectors φ_k, we obtain the following Fourier representation of the energy:

$$E(t) = \sum_{k=1}^{\infty} e_k(t), \text{ with } e_k(t) = \frac{1}{2}\left[|u_k'(t)|^2 + \lambda_k |u_k(t)|^2\right].$$

Each e_k satisfies the decay property:

$$e_k(t) \leqslant C e_k(0) e^{-\omega_k t},$$

where $C > 0$ does not depend on k and on the data (u^0, u^1) and ω_k is the exponential decay rate of the k-th Fourier component:

$$\omega_k = \begin{cases} \frac{a - \sqrt{a^2 - 4\lambda_k}}{2}, & a^2 > 4\lambda_k \\ \frac{a}{2}, & a^2 < 4\lambda_k. \end{cases}$$

When $a^2 = 4\lambda_k$, the decay law of the Fourier component is

$$e_k(t) \leqslant C e_k(0) t \exp(-\omega_k t),$$

with $\omega_k = a/2$.

The exponential decay of the energy

In conclusion,

$$E(t) \leqslant CE(0)\exp(-\omega t),$$

with

$$\omega = \omega(a) := \begin{cases} \frac{a}{2} & a^2 < 4\lambda_1, \\ \frac{a - \sqrt{a^2 - 4\lambda_1}}{2} & a^2 > 4\lambda_1, \end{cases}$$

When $a^2 = 4\lambda_1$, the decay result of the energy is slightly different:

$$E(t) \leqslant CE(0)t\exp\left(-\frac{a}{2}t\right).$$

The function $\omega(a)$ has the following monotonicity properties:

- linearly increasing for $a \in [0, 2\sqrt{\lambda_1}]$.
- decreasing for $a > 2\sqrt{\lambda_1}$.
- $\omega(a) \to 0$ as $a \to \infty$.
- The maximal decay rate is attained when $a = 2\sqrt{\lambda_1} \Rightarrow$ overdamping

When the initial data (u^0, u^1) involves only high frequency Fourier components for which $4\lambda_k \geqslant a^2$, then the exponential decay rate is simply $a/2$.

Remedies for the overdamping

For the heat equation with potential

$$\begin{cases} u_t - \Delta u + au = 0 & \Omega \times (0,\infty), \\ u = 0 & \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \Omega, \end{cases} \tag{25}$$

the decay of the energy is exponential for any $a > 0$

$$||u(t)||_{L^2(\Omega)} \leqslant \exp\Big(-\frac{(\lambda_1 + a)}{2} t \Big) ||u^0||_{L^2(\Omega)}.$$

For the heat equation there is no overdamping since it is a first-order equation in time, while the wave equation is second-order in time, having two unknowns, u and u_t:

$$u_t = v, \quad v_t = \Delta u - av.$$

The fact that there is only one dissipative potential in the second equation produces overdamping. Remedy: Use two potentials $a > 0$ and $b > 0$ affecting both u_t and u:

$$\begin{cases} u_{tt} - \Delta u + au_t + bu = 0 & \Omega \times (0,\infty) \\ u = 0 & \partial\Omega \times (0,\infty) \\ u(0) = u^0, \; u_t(0) = u^1 & \Omega. \end{cases} \tag{26}$$

The energy is given by

$$E_b(t) = \frac{1}{2} \int_\Omega \Big[|u_t(x,t)|^2 + |\nabla u(x,t)|^2 + bu^2(x,t) \Big] \, dx$$

and satisfies the decay property $\frac{dE_b}{dt}(t) = -a \int_\Omega u_t^2(x,t)\, dx$.

The Fourier coefficients $u_k(t)$ in (23) satisfy the damped harmonic oscillator:

$$u_k''(t) + (\lambda_k + b)u_k(t) + au_k'(t) = 0.$$

In this case, the roots of the characteristic polynomial are of the form:

$$\mu_{\pm}^k = \frac{-a \pm \sqrt{a^2 - 4(\lambda_k + b)}}{2}.$$

For any $a > 0$, we can choose $b > 0$ sufficiently large so that

$$a^2 < 4(\lambda_1 + b).$$

In that case, each Fourier component decay exponentially at rate $-a/2$. Then

$$E_b(t) \leqslant CE_b(0) \exp\Big(-\frac{a}{2}t\Big).$$

Similar analysis when passing to the limit as $\epsilon \to 0$ from the dissipative wave equation

$$\varepsilon u_{tt} - \Delta u + u_t = 0$$

to the heat equation

$$u_t - \Delta u = 0.$$

Semigroup theory for the wave equation

The initial boundary value problem for the wave equation in the open bounded domain $\Omega \subset \mathbb{R}^n$

$$\left\{ \begin{array}{cc} u_{tt} - \Delta u = 0 & \Omega \times (0, \infty) \\ u = 0 & \partial\Omega \times (0, \infty) \\ u(x, 0) = u^0, \quad u_t(x, 0) = u^1(x) & \Omega \end{array} \right. \tag{27}$$

To apply the semigroup theory, it is convenient to write (27) as a first-order system:

$$\left\{ \begin{array}{ccc} u_t & = & v \\ v_t & = & \Delta u. \end{array} \right. \tag{28}$$

The unknown contains two components, position and velocity: $U = (u, v) = (u, u_t)$.
With this notation, U verifies the system

$$U_t = AU, \quad U(0) = U^0 := (u^0, u^1), \tag{29}$$

where A is the linear operator

$$A := \left(\begin{array}{cc} 0 & I \\ \Delta & 0 \end{array} \right).$$

The natural space to solve the wave equation (28) is the energy space $H := H_0^1(\Omega) \times L^2(\Omega)$ since:

- it is the space where the energy below is conserved in time

$$E(t) := \frac{1}{2} \int_\Omega \left[|\nabla u(x, t)|^2 + |u_t(x, t)|^2 \right] dx$$

- the boundary condition $u = 0$ in $\partial\Omega$ requires $u \in H^1(\Omega)$ vanishing on $\partial\Omega$, i.e. $u \in H_0^1(\Omega)$.

The domain of the operator A

The norm in H is

$$||(f,g)||_H := \left[||f||^2_{H^1_0(\Omega)} + ||g||^2_{L^2(\Omega)}\right]^{1/2},$$

where

$$||f||_{H^1_0(\Omega)} = \left[\int_\Omega |\nabla f|^2\, dx\right]^{1/2} \text{ and } ||g||_{L^2(\Omega)} = \left[\int_\Omega |g|^2\, dx\right]^{1/2}.$$

The domain $D(A) \subset H$ of the operator A is defined as $D(A) := \{V \in H \text{ s.t. } AV \in H\}$.
A is a linear unbounded operator $A : D(A) \to H$. More precisely,

$$D(A) := \{(u,v) \in H^1_0(\Omega) \times L^2(\Omega) \text{ s.t. } v \in H^1_0(\Omega) \text{ and } \Delta u \in L^2(\Omega)\}$$

$$\text{or } D(A) := \{(u,v) \in H^1_0(\Omega) \times H^1_0(\Omega) \text{ s.t. } \Delta u \in L^2(\Omega)\}.$$

When Ω is of class C^2 or a convex domain, the classical result of elliptic regularity guarantees that $D(A) = \left[H^2(\Omega) \cap H^1_0(\Omega)\right] \times H^1_0(\Omega)$.

The operator A is anti-adjoint, i.e. $A^* = -A$.

To prove this, we use the fact that the Laplace operator $\Delta : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ is anti-adjoint. Indeed, for any $U,\ \widetilde{U} \in D(A)$,

$$\begin{aligned}\left(AU, \widetilde{U}\right)_H &= \left(v, \widetilde{u}\right)_{H^1_0(\Omega)} + \left(\Delta u,\ \widetilde{v}\right)_{L^2(\Omega)} \\ &= \int_\Omega \left[\nabla v \cdot \nabla \widetilde{u} + \Delta u \widetilde{v}\right] dx = -\int_\Omega \left[v \Delta \widetilde{u} + \nabla u \cdot \nabla \widetilde{v}\right] dx \\ &= -\left(U, A\widetilde{U}\right)_H.\end{aligned}$$

Hille-Yosida Theorem

Types of solutions for the abstract equation (29)

- strong solutions, i.e. $U \in C([0,\infty); D(A)) \cap C^1([0,\infty); H)$. For the wave equation, this means $u \in C([0,\infty); H^2 \cap H_0^1(\Omega)) \cap C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega))$
- weak solutions which are less regular, in the sense of distributions, i.e. $U \in C([0,\infty); H)$ or, for the wave equation $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$. Since u is solution of the wave equation, it has the additional regularity $u \in C^2([0,\infty); H^{-1}(\Omega))$.
- ultra-weak solutions are much less regular. For the wave equation, they belong to the class $u \in C([0,\infty); L^2(\Omega)) \cap C^1([0,\infty); H^{-1}(\Omega)) \cap C^2([0,\infty); [H^2 \cap H_0^1(\Omega)]')$.

Definition

A linear and unbounded operator $A : D(A) \subset H \to H$ is called

- dissipative if $(AU, U)_H \leqslant 0$ for all $U \in D(A)$.
- maximal dissipative if, moreover, it satisfies
  $R(I - A) = H \Leftrightarrow \forall F \in H,\ \exists U \in D(A)$ s.t. $U - AU = F$.

Theorem (Hille-Yosida Theorem)

Let A be a maximal dissipative operator in a Hilbert space H. Then, for any $U^0 \in D(A)$, there exists an unique solution of (29), $U \in C([0,\infty); D(A)) \cap C^1([0,\infty); H)$. Moreover, for any $t > 0$,

$$||U(t)||_H \leqslant ||U^0||_H \text{ and } \left|\left|\frac{dU}{dt}(t)\right|\right|_H = ||AU(t)||_H \leqslant ||AU^0||_H.$$

Strong and weak solutions for the wave equation

Strong solutions: $H := H_0^1 \times L^2(\Omega)$, $D(A) := [H^2 \cap H_0^1] \times H_0^1(\Omega)$

- A is dissipative. Since A is anti-adjoint, $(AU, U)_H = -(AU, U)_H$ and then $(AU, U)_H = 0$ for any $U \in D(A)$. In fact, when $(AU, U)_H = 0$, the energy $||U(t)||_H^2/2$ of the solution of (29) is conserved in time.
- A is maximal. For any $F = (f, g) \in H$ i.e. $f \in H_0^1(\Omega)$ and $g \in L^2(\Omega)$, there exists at least a solution $U = (u, v) \in D(A) = (H^2 \cap H_0^1) \times H_0^1(\Omega)$ of $(I - A)U = F$. More precisely, (u, v) satisfy $u - v = f$ and $v - \Delta u = g$. Since $v = u - f$, then

$$u - \Delta u = g + f, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega. \tag{30}$$

  Since $f + g \in L^2(\Omega)$, classical results of existence, uniqueness and regularity results for the Dirichlet problem (30) guarantee that (30) has an unique solution $u \in H^2 \cap H_0^1(\Omega)$. Since $f \in H_0^1(\Omega)$ and $u \in H^2 \cap H_0^1(\Omega)$, then $v = u - f \in H_0^1(\Omega)$.

Theorem

If Ω is a bounded domain of class C^2, for any data $(u^0, u^1) \in [H^2 \cap H_0^1(\Omega)] \times H_0^1(\Omega)$, (27) has an unique solution $u \in C([0, \infty); H^2 \cap H_0^1(\Omega)) \cap C^1([0, \infty); H_0^1(\Omega)) \cap C^2([0, \infty); L^2(\Omega))$.

Weak solutions: $H := L^2 \times H^{-1}(\Omega)$, $D(A) := H_0^1 \times L^2(\Omega)$

Theorem

If Ω is a bounded domain of class C^2, for any data $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, the wave equation (27) has an unique solution $u \in C([0, \infty); H_0^1(\Omega) \cap C^1([0, \infty); L^2(\Omega)) \cap C^2([0, \infty); H^{-1}(\Omega))$.

Obtain weak solution from strong solutions

Suppose $U(t)$ is the solution of (29) with data $U^0 \in H$. Then $U(t) = U^0 + A\int_0^t U(s)\,ds$.

Set $V(t) = V^0 + \int_0^t U(s)\,ds$.

Then $V_t = AV + U^0 - AV^0, \quad V(0) = V^0$.

If $AV^0 = U^0$ would have an unique solution $V^0 \in D(A)$, then $V(t)$ would be the solution of (29) with data $V^0 \in D(A)$.

According to Hille-Yosida Theorem, (29) with initial data $V^0 \in D(A)$ has an unique solution $V \in C([0,\infty); D(A)) \cap C^1([0,\infty); H)$ and then $U = V_t \in C([0,\infty); H)$.

Given $U^0 \in H$, does there exist an unique solution $V^0 \in D(A)$ of $AV^0 = U^0$? Maximality does not guarantee this, since only $I - A$ is a full rank operator.

Set $W(t) = \exp(\lambda t)U(t)$, which solves the equation

$$W_t = AW + \lambda W,\ t > 0, \quad W(0) = U^0. \tag{31}$$

U is a weak (strong) solution of (29) iff W is a weak (strong) solution of (31).

Set $\lambda = -1$ and $\widetilde{V}(t) = \widetilde{V}^0 + \int_0^t W(s)\,ds$.

Then $\widetilde{V}_t = A\widetilde{V} + U^0 - (A\widetilde{V}^0 - \widetilde{V}^0), \quad \widetilde{V}(0) = \widetilde{V}^0$.

If $(A - I)\widetilde{V}^0 = U^0$ would have an unique solution $\widetilde{V}^0 \in D(A)$, then $\widetilde{V}(t)$ would be the solution of (29) with data $\widetilde{V}^0 \in D(A)$. But now $A - I$ is a full rank operator, so that for any $U^0 \in H$, the equation $(A - I)\widetilde{V}^0 = U^0$ has an unique solution.

Generator of a semigroup

Connection between the semigroups and Fourier theories

- Strong solutions: $D(A) = [H^2 \cap H_0^1] \times H_0^1 = \left\{ (a_k, b_k)_k \text{ s.t. } \sum_{k=1}^{\infty} [\lambda_k^2 |a_k|^2 + \lambda_k |b_k|^2] < \infty \right\}.$
- Weak solutions: $D(A) = H_0^1 \times L^2 = \left\{ (a_k, b_k)_k \text{ s.t. } \sum_{k=1}^{\infty} [\lambda_k |a_k|^2 + |b_k|^2] < \infty \right\}$
- Ultra-weak solutions: $D(A) = L^2 \times H^{-1} = \left\{ (a_k, b_k)_k \text{ s.t. } \sum_{k=1}^{\infty} [|a_k|^2 + \lambda_k^{-1} |b_k|^2] < \infty \right\}.$

When A is a maximal-dissipative operator, it is the generator of a semigroup $S(t) : H \to H$ associating to any $U^0 \in H$, the solution $U(t) = S(t)U^0 = \exp(At)U^0$ of (29) at time $t > 0$.

The semigroup $\{S(t)\}_{t \geqslant 0} = \{\exp(At)\}_{t \geqslant 0}$ is an one-parameter family of linear bounded operators.

The semigroup $S(t)$ generated by a maximal-dissipative operator A is a contraction, for any $t > 0$.

Moreover, any semigroup $S(t)$ verifies the following properties:

- $S(0) = I$,
- $t \to S(t)U^0$ is continuous from $[0, \infty)$ in H for any $U^0 \in H$
- $S(t) \circ S(s) = S(t + s)$.

Semigroup theory applied to variable coefficients, nonlinear equations

Consider the non-homogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = f & (x,t) \in \Omega \times (0,\infty) \\ u = 0 & (x,t) \in \partial\Omega \times (0,\infty) \\ u(0) = u^0, \;\; u_t(0) = u^1 & x \in \Omega. \end{cases} \tag{32}$$

Examples:

- $f = f(x,t)$ an external force;
- $f = p(x,t)u(x,t) \Rightarrow$ wave equation with potential;
- $f = f(u) \Rightarrow$ semilinear wave equation; e.g. $f(u) = |u|^{p-1}u$.

Problem (32) can be written as $u_t = v, \; v_t = \Delta u + f$ or, in a more abstract form $(F = (0, \; f))$

$$U_t = AU + F, \;\; t > 0, \quad U(0) = U^0. \tag{33}$$

(33) can be solved by the variation of constant formula:

$$U(t) = S(t)U^0 + \int_0^t S(t-s)F(s)ds. \tag{34}$$

- $F \in L^2(0,T;D(A)) \Rightarrow S(t-s)F(s) \in L^1(0,t;D(A))$. Indeed, Hille-Yosida Theorem implies $||S(t-s)F(s)||_H \leqslant ||F(s)||_H$ and $AS(t-s)F(s)||_H \leqslant ||AF(s)||_H$. Thus
$$\int_0^t S(t-s)F(s)\,ds \in C([0,T];D(A)).$$
- (33) has a strong solution $U \in C([0,\infty);D(A)) \cap C^1([0,\infty);H)$ if $F \in C([0,T];H)$ implying
$$\int_0^t S(t-s)F(s)\,ds \in C^1([0,T];H).$$

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