An Algorithm for Density

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April 22, 2020

Outline

QUANTIFYING DENSITY

Assume $L : H \to H$ is a linear, bounded operator with dense range. Then, for all $f \in H$ and $\varepsilon > 0$ there exists $u \in H$ such

$$||Lu - f||_H \leq \varepsilon. \tag{1}$$

Of course the density of the range often happens without the map being surjective. This occurs frequently when looking to the evolution of time-irreversible semigroups and is relevant in control problems (*the system can be steered to a dense set of targets but not to all targets*).

Example: $Lu = G * u$, G being a gaussian.

In practice it is important to have a methodology/algorithm to build the solution u to (1).

Note that, according to Hahn-Banach Theorem, the rank of L is dense if and only if L^*, its adjoint, is injective: $L^* v = 0$ implies $v = 0$.

Consider now the functional:[1]

$$J(v) = \frac{1}{2}||L^* v||_H^2 + \varepsilon||v||_H - (f, v)_H.$$

Note that both f and ε in the density property are involved in this definition of J.

If, in addition to the injectivity property, we had,

$$||L^* v||_H^2 \geq \alpha||v||_H^2,$$

then the functional J would be coercive even for $\varepsilon = 0$. But the term added by means of $\varepsilon > 0$ is needed to ensure coercivity under the sole assumption that L^* is injective.

If J achieves its minimum at $\tilde{v}$, then considering that $J(\tilde{v} \pm \delta v) - J(\tilde{v}) \geq 0$, we obtain by considering the leading term when $\delta \to 0$:

$$|(L^*(\tilde{v}), L^* v)_H - (f, v)_H| \leq \varepsilon||v||_H.$$

i. e.

$$|(LL^*(\tilde{v}) - f, v)_H| \leq \varepsilon||v||_H, \text{ i. e., } ||LL^*(\tilde{v}) - f||_H \leq \varepsilon.$$

This means that $u = L^*(\tilde{v})$ is the solution we were looking for.

[1] An interesting exercise is to consider the functional $\tilde{J}(v) = \frac{1}{2}||L^* v||_H^2 + \varepsilon||v||_H^2 - (f, v)_H$ and see what happens.

Does the minimizer of J exist?

$$J(v) = \frac{1}{2}||L^* v||_H^2 + \varepsilon ||v||_H - (f, v)_H.$$

$J : H \to \mathbf{R}$ is continuous and convex in a Hilbert space. It suffices to show coercivity.

We claim that, under the density assumption, or the injectivity of L^*, the functional is coercive in the sense that

$$\underline{\lim}_{||v||_H \to \infty} J(v)/||v||_H \geq \varepsilon.$$

Set $v_j : ||v_j||_H \to \infty$. Normalizing things: $\hat{v}_j = v_j / ||v_j||_H$ and then

$$J(v_j)/||v_j||_H = \frac{1}{2}||v_j||_H ||L^* \hat{v}_j||_H^2 + \varepsilon - (f, \hat{v}_j)_H.$$

The delicate case is when $||L^* \hat{v}_j||_H \to 0$. Then, in the limit, $L^* \hat{v} = 0$ which implies $\hat{v} = 0$. This implies weak convergence to zero and thus $(f, \hat{v}_j)_H \to 0$. Consequently,

$$J(v_j)/||v_j||_H \geq \varepsilon - (f, \hat{v}_j)_H \to \varepsilon.$$

Assume now that E is a finite-dimensional subspace of H. Then, for all $f \in H$ and $\varepsilon > 0$ one can find $u \in H$ such that

$$||Lu - f||_H \leq \varepsilon; \quad \pi_E Lu = \pi_E f.$$

Proof: Minimize

$$J(v) = \frac{1}{2}||L^* v||_H^2 + \varepsilon||(1 - \pi_E)v||_H - (f, v)_H.$$

J. L. LIONS & E. ZUAZUA. The cost of controlling unstable systems: The case of boundary controls. J. Anal. Mathématique, LXXIII (1997), 225-249.