Acta Math. Univ. Comenianae
Vol. LXXXII, 1 (2013), pp. 125{146

BLOWUP FOR A TIME-OSCILLATING NONLINEAR HEAT EQUATION

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Abstract. In this paper, we study a nonlinear heat equation with a periodic time-oscillating term in factor of the nonlinearity. In particular, we give examples showing how the behavior of the solution can drastically change according to both the frequency of the oscillating factor and the size of the initial value.

1. Introduction

Let be a smooth, bounded domain of R^N and x > 0. Let > 0 and let 2 C(R; R

Moreover, if the solution of the limiting equation is global and decays (in an appropriate sense) as $t \to \infty$, then the solution of the time-oscillating equation is also global for $|\omega|$ large. It is natural to expect that if the solution of the limiting equation blows up in finite time, then so does the solution of the time-oscillating equation for $|\omega|$ large, but this question seems to be open. (See [3, Question 1.7].)

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instructive to consider the ODE associated with the heat equation (1.1), i.e.,

$$v' + av = (\omega t)|v| v; \tag{1.4}$$

where $a > 0$ and the limiting ODE

$$V' + aV = A( )|V| V. \tag{1.5}$$

The solutions v of (1.4) and V of (1.5) with the initial conditions $v(0) = V(0) = x > 0$ are given by

$$v(t) = e^{at}(x \quad h(t; \omega))^{\frac{1}{}}; \tag{1.6}$$

where

$$h(t; \omega) = \int_0^t e^{a s} (\omega s)\, ds; \tag{1.7}$$

and

$$V(t) = e^{at}(x \quad a^{1}A( )[1 \quad e^{a t}]) \tag{1.8}$$

The solution V blows up in nite time if and only if $x < \frac{1}{a}A( )$. For such

Open problem 1.4. Let > 0, let 2 C(R; R) be -periodic and let A( ) be de ned by (1.3). Assume A( ) > 0 and let 2 C_0( ) be such that the corresponding solution of (1.2) blows up in nite time. Does the solution of (1.1)

Theorem 1.6. There exist >*AT E*

the convergence results (Propositions 1.1 and 1.2), while Theorems 1.6, 1.7 and 1.3 are proved in Sections 4, 5, and 6 respectively. The last section of the paper is an appendix devoted to an abstract result which we use in the proof ot Theorem 1.3.

Notation

We denote by

for all $0 \quad t < T$. In addition, if $v_0; w_0$ both satisfy (2.6) and $v; w$ are the

Lemma 2.4. Let *f* 2 L^{1}(0; *1*), *f* 0 and let *v* 2 *C*([0; $T_{\max}$) 9.9626 Tf 11.623 0 let*max*

Remark 2.7. We claim that there exist an initial value and a function as in Problem 1.5 such that the solution of (1.1) with ! = 1 blows up in nite time after picking up negative values of . To see this, let 2 C_c^1 ( ), 0051 6 0 and let be the solution of

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= in ;
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It follows from the strong maximum principle that a ' 1

Given any $2\ C_0(\ )$, let u be the corresponding solution of (1.1) with $! = 1$. It follows easily by comparison with the solution $(\ t)\ \frac{1}{}$

Proof of Proposition 1:1. Fix $0 < T < T_{\max}$ and set

$$M = 2$$


6

Applying Remark 2.5 (with $f(t)$ $\quad$ $(!t)$), we conclude by using (2.12) that u

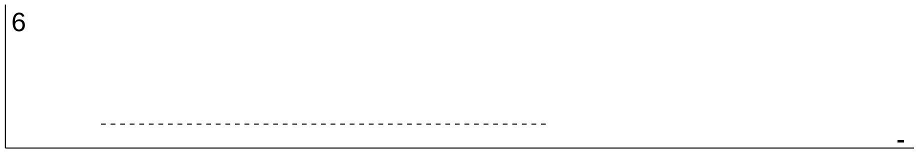
Step 8. Conclusion. Property (i) follows from Step 1. Property (ii) follows from Steps 2 and 3. Property (iii) follows from Steps 4, 5 and 6. Property (iv) follows from Step 7.

5. Proof of Theorem 1.7

We consider , $k_0; k_1; k_2; k_3$ as in the preceding section and we let

$$(t) \qquad (3 \quad t):$$

(See Figure 4.) Given $k > 0$ and $! > 0$, we consider the solution $u_{k;!}$ of (1.1) with



Step 5. Conclusion. Property (ii) follows from Steps 1, 2 and 3. Property (iii) follows from Steps 1 and 4. (Note that the map = 1 22.597 Td [(6.)]TJ/F43 9.9626 Tf 12.73 0 Td [(Pr)22(oof)-377(of)-378(Theorem)]T

ly Lipschitz $L^2(\ )\ L^2$

Remark 6.2. Notr that wr could obtain (with thr samr proof) conclusions similar to those of Theorem 1.3 for equations slightly more general than (1.9). For example, one could replace thr nonlinearity $f(u) = kuk_{L^2}u$ in (1.9) by thr more general one so that 0 1. Let 0, set = ′ and let u be the solution of (A.2) with initial value

On the other hand, it follows from Proposition A.3 that

$$u\ (t) = \frac{\|u\ (t)\|}{\|e^{tL}(\ \ ')\|}$$