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Numerics for the Control of Partial Differential Equations

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Without Abstract

Introduction

Control theory is now an old subject. It emerged with the Industrial Revolution and has been continuously evolving since. New technological and industrial processes and mechanisms need new control strategies, and this leads to new Mathematics of Control as well. At present control theory is certainly one of the most interdisciplinary areas of research, and it arises vigorously in most modern applications.

Since its origins (see [*3*, *12*]) the field has evolved tremendously, and different tools have been developed to face the main challenges that require to deal with a variety of models: Ordinary Differential Equations/Partial Differential Equations, Linear/Nonlinear, Deterministic/Stochastic, etc.

Practical control problems can be formulated in many different ways, requiring different kinds of answers, related to the different notions of control; the various possible modeling paradigms; and the degree of precision of the result one is looking for optimal control, controllability, stabilizability, open-loop versus feedback or close-loop controls, etc. Last but no least, the practical feasibility and implementability of the control mechanisms that theory produces needs to be taken into account.

In this multifold task the mathematical theory of control that has been developed is nowadays a rich combination of, among other fields, Fourier, Functional, Complex and Stochastic Analysis, ODE and PDE theory and Geometry (see [*8*, *25*]).

Needless to say, in practice, controls need to be computed and implemented through numerical algorithms and simulations. Numerical analysis is then necessary to design convergent algorithms allowing for an efficient approximation and computation of controls. Again, the existing theory on numerical methods for control is wide and the employed techniques diverse, adapted to the different problems and contexts mentioned above.

In this article we present a partial panorama of the state of the art in what concerns numerical methods for solving control problems for partial differential equations. This article cannot be exhaustive. We have chosen to focus on a specific topic that we consider to play a central role in the theory. We also take the opportunity to point towards some other related issues of current and future research.

Problem Formulation

Optimal controls for PDEs can be often characterized as the solutions of an optimality system coupling the state to be controlled and the adjoint state. One can then numerically approximate these systems to get a numerical approximation of the control. This leads to the so-called *continuous approach* in which one first develops the control theory at the level of the continuous models (PDEs) and then uses numerical analysis for approximating the control. The *discrete approach* consists roughly on proceeding all the way around: We first discretize the PDEs and then use finite-dimensional control theory to compute the controls of the discretized model. In the last few years, it has been clearly understood that the two approaches do not necessarily lead to the same results and, in particular, that the convergence of the procedures is not ensured by the fact of having used a convergent numerical approximation for the underlying PDE dynamics and the control requirement. In fact, each of the approaches has its advantages and drawbacks. In particular, as analyzed in [*10*, *11*] in detail:

- The continuous approach may diverge if one mimics at the discrete level in a straightforward manner iterative algorithms that, at the continuous one, lead to the right optimal control characterized by the optimality system.
- The discrete approach may diverge since the controls for the discrete dynamics do not necessarily converge to those of the continuous dynamics as the mesh-size parameter tends to zero.

In both cases the reason for these divergence phenomena is the same: the presence of high-frequency numerical oscillations that do not reproduce the propagation properties of continuous wave equations and that eventually leads to the failure of convergence of the controls of the discrete dynamics to those of the continuous one. This makes the discrete approach fail. But, for the same reason, the continuous approach may fail as well. Indeed, when implementing at the discrete level the iterative methods developed to compute the control of the continuous one, one is eventually led to the control of the discrete dynamics which, as mentioned above, does not necessarily converge to the continuous one. The same occurs to other methods, based on different iterative algorithms for building continuous controls, as for instance, the one developed in [*7*] which implements D. Russell’s method of “stabilization implies control” (see [*22*]).

Similarly the cure is also the same in both cases: filtering the high frequencies so to concentrate the energy of numerical solutions in the low-frequency components that behave truly as continuous waves. The need of this high frequency filters was already pointed out by R. Glowinski, J. L. Lions, and collaborators (see, for example, [*13*]).

The simplest and most paradigmatic example of those pathologies is the wave equation. Indeed, the control of the discrete dynamics generated by convergent numerical schemes of a $1-D$ wave equation can dramatically diverge as the mesh size tends to zero even in situations where the wave equation itself is easily controllable (see [*24*]). This is due to the pathological behavior of the high-frequency numerical solutions. Indeed, while solutions of the continuous wave equation propagate with velocity equal to one, solutions of most numerical schemes can propagate with an asymptotically (as the mesh-size parameter tends to zero) small *group velocity* [*23*]. Furthermore, for the continuous wave equation, the fact that all waves propagate with the same velocity reaching the control region (for instance, the boundary of the domain) in a uniform time is the reason why controllability holds. Similarly, the very slow propagation of the very high-frequency numerical wave packets is the reason why the controls of the numerical scheme may diverge, even with an exponential rate, as the mesh-size parameters tend to zero.

The link between velocity of propagation of solutions of wavelike equations and the boundary control properties of these processes is rigorously established through the so-called Geometric Control Condition (GCC) [*1*] which ensures, roughly, that wavelike equations are controllable if and only if all rays of Geometric Optics enter the control region in an uniform time.

From a numerical analysis viewpoint, although the existing theory is rather complete for constant coefficient wave equations in uniform numerical grids in which the Fourier representation of solutions is available, plenty is still to be done for dealing with general variable coefficient wave equations discretized in nonuniform grids. When the grid can be mapped smoothly into a uniform one, the corresponding analysis will need of microlocal and Wigner measures tools.

Related Issues and Perspectives

There are other topics arising in the intersection of the theory of PDEs and numerical analysis and in which similar issues appear. Important progress has been done recently developing ideas that are closely related to the ones discussed above and in which a careful comparison of continuous versus discrete methods is necessary. We mention here some of them with some basic related bibliography. Neither the list of topics nor that of the main related references is complete.

- *Filtering*: As mentioned above, the most natural cure for the high-frequency numerical pathologies is filtering. This can be done in various different manners: by using some Fourier filtering mechanism [*24*], adding numerical artificial viscosity terms [*15*], wavelet decompositions [*19*] or; the most frequent one, easy to implement, a two-grid algorithm originally introduced by R. Glowinski (see [*14*] and references therein). This leads to numerical algorithms for computing the controls that actually converge but at the prize of relaxing the control requirement. Indeed, when filtering the numerical solutions, one ends up controlling not the whole solution of the numerical scheme but rather a low-frequency projection. A more systematic study of the filtering mechanisms on nonuniform grids and the related adaptivity techniques (depending on the data to be controlled, according to the time evolution of controlled solutions) is still to be developed.

- *Feedback stabilization of wave processes*: Similar issues arise in the context of the exponential stabilization of wave equations by means of feedback mechanisms. For the continuous wave equation, this issue is well understood, and the exponential decay is guaranteed provided the feedback is effective in a subset of the domain satisfying the GCC. But, as in the context of controllability, the decay rate fails to be uniform when the PDE is replaced by a numerical approximation scheme, and this is due, again, to the high-frequency spurious solutions. Extra artificial viscous damping is then required in order to ensure the uniform exponential decay of solutions (see [9] and the references therein).
- *Optimal design of flexible structures*: The subject of the optimal design of controllers and actuators for systems governed by PDEs is also widely open. Again, the issue of whether the discrete approach suffices to compute accurate approximations of continuous optimal shapes and designs is a relevant and widely open issue. But, in this context, theory is still lacking of completeness. This is even the case at the level of the continuous problem in which the existence and geometric properties of optimal shapes and designs are often unknown. For the problem of optimal placement of observers and actuators for models of vibrations, we refer to [21] and the references therein. We also refer to [4, 6] where, in a number of $1-d$ and $2-d$ time-independent model examples, the convergence of the discrete optimal shapes towards the continuous ones is proved.
- *Optimal design in fluid mechanics in the presence of shocks*: The debate on whether one should develop either continuous or discrete methods for solving optimal control and design problems for PDEs has been also very intense as is still ongoing in the context of fluid mechanics, motivated by optimal design in aerodynamics. This issue is particularly important when solutions develop shock discontinuities, as it happens for some of the most relevant models consisting on scalar conservation laws or hyperbolic systems. Because of the discontinuity of solutions, classical linearizations are not justified and an ad hoc linearization is required, taking care of the Rankine-Hugoniot condition. This allows to derive not only the sensitivity of the smooth components of solutions but also of the shock location. In this context a straightforward linearization of the discrete models does not necessarily lead to the correct sensitivity analysis of the continuous ones. In view of this, the sensitivity of shocks has to be carefully incorporated to the numerical methods aiming to approximate the optimal controls and shapes. We refer to [5] where a hybrid method is proposed, alternating the continuous and the discrete approaches in the implementation of

descent methods for an inverse design problem associated with the inviscid Burgers equation.

- *Inverse problems*: Similar issues arise in the context of inverse problems for wavelike problems and the classical Calderón's problem. In recent years a number of works have been devoted to adapt the techniques for an efficient numerical approximation of the controls of the wave equation to inverse problems. We refer for instance to [2] where this has been done in the context of the problem of recovering the potential of a $1 - d$ wave equation from one measurement by means of finite-difference schemes adding a Tychonoff regularization term.
- *The heat equation*: There is also a wide literature on the null control of heat equations, which consists on driving the solutions to the zero rest by means of a localized control. Null controllability turns out to be equivalent to an observability inequality for the adjoint heat equation, a fact that is by now well known to hold in an arbitrarily small time and from arbitrary open nonempty observation subsets. These inequalities have been established using Fourier series arguments in $1 - d$ and Carleman inequalities in the multi-d case.

  Much less is known from the numerical analysis point of view. Of course, in this context of the heat equation, both the continuous and the discrete approach can be implemented as well. In [18] a numerical method is derived which combines the efficient numerical algorithms for the control of the wave equation that, in particular, uses the filtering of the numerical high frequencies and the Kannai transform that allows transmuting control properties of the wave equation into the heat one [16]. In this way one can derive a performant method for computing numerical approximations of the controls, avoiding the classical ill-posedness of the problem, related to the strong time irreversibility of the heat equation. Note however that the controls obtained in this way are not those of minimal L^2-norm.

  Another important development in this context is related to the Carleman inequalities for discrete approximations of the spectrum of elliptic equations and the heat equation. This allows proving a number of results on the uniform control of numerical approximation schemes for linear and semilinear heat equations. Note however that the filtering of high frequencies is needed because of the remainder terms that the discrete Carleman inequalities exhibit with respect to the continuous one. But this does not arise because of technical reasons only. In fact, as indicated in [25], in the multidimensional case, the standard unique continuation properties of the eigenfunctions of the Laplacian and the heat equation do not hold for finite-difference approximations at high frequencies. Thus, the filtering of high-frequency numerical components is a must for multi- d problems.

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