Internal controllability for parabolic systems involving analytic non-local terms

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Abstract

We deal with the problem of internal controllability of a system of heat equations posed on a bounded domain with Dirichlet boundary conditions and perturbed with analytic non-local coupling terms. Each component of the system may be controlled in a different subdomain. Assuming that the unperturbed system is controllable -a property that has been recently characterized in terms of a Kalman-like rank condition-, we give a necessary and sufficient condition for the controllability of the coupled system under the form of a unique continuation property for the corresponding elliptic eigenvalue system. The proof relies on a compactness-uniqueness argument, which is quite unusual in the context of parabolic systems, previously developed for scalar parabolic equations. Our general result is illustrated by two simple examples.

Keyworlds: parabolic systems; non-local potentials; null controllability; Kalman rank condition; spectral unique continuation.

MSC: 35K40; 93B05; 93B07.

1 Introduction

1.1 Motivation

Nonlocal parabolic systems are relevant in a variety of applications to Biology and Physics, see for instance [25]. They have been analyzed exhaustively in the recent past, in particular in the context of the non-local fractional Laplacian, and significant progress has been achieved. But controllability issues for these models remain very much unexplored. Here we analyse parabolic systems coupled by non-local lower order perturbations, the principal part being a classical constant coefficient parabolic system.

The content of this paper is a natural combination of the methods developed in [22] to achieve sharp results for parabolic systems coupled through constant coefficient matrices and those in [17] devoted to scalar equations perturbed by non-local lower order potentials. Our goal here is to derive a simple and exploitable spectral necessary and sufficient condition of controllability and the corresponding dual observability one.

This paper is very much inspired in the pioneering ideas introduced by J. L. Lions in his famous SIAM Review article [21] that stimulated a significant step forward on the state of the art. The early developments in this field were summarized with mastery in the celebrated survey article by D. L. Russell [26]. The presentation in this paper is concise, relying significantly on various tools of Functional Analysis

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that are developed and presented in a self-contained manner in the more recent book by Phillippe G. Ciarlet [10]. *This article is dedicated to him in the occasion of his 80th birthday with gratitude and admiration for his mastery and continuous support. Merci Philippe!*

1.2 Problem formulation and main result

Let us now present the problem under consideration in more detail.

Let Ω be a smooth domain of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$), $T > 0$, $n \in \mathbb{N}^*$ and $m \in \mathbb{N}^*$ (with possibly $m < n$). Let ω_i ($i \in [|1, m|]$) be some open subsets of Ω that can be chosen arbitrarily (in particular all the ω_i's may be disjoint).

We are interested in the controllability of the following system of heat equations with Dirichlet boundary conditions

$$\begin{cases} \partial_t Y = D\Delta Y + \displaystyle\int_\Omega A(x,\xi) Y(t,\xi) d\xi + \sum_{i=1}^m B_i u_i 1_{\omega_i} & \text{in} \quad (0,T)\times\Omega, \\ Y(t,x) = 0 \quad \text{in} \quad (0,T)\times\partial\Omega, \\ Y(0) = Y^0, \end{cases} \tag{1.1}$$

with $Y^0 \in [L^2(\Omega)]^n$, $u = (u_1, \dots u_m) \in [L^2(\Omega)]^m$ (which play the role of distributed controls) , $A \in \mathcal{M}_n(\mathbf{H}) \subset \mathcal{M}_n(L^2(\Omega\times\Omega))$ (where $\mathbf{H}$ is a space of admissible potentials that will be introduced afterwards in (1.11)), B_i is the i-th column of $B \in \mathcal{M}_{n,m}(\mathbb{R})$.

The coupling matrix $D \in \mathcal{M}_n(\mathbb{R})$ is assumed to satisfy the ellipticity condition

$$\langle DX, X\rangle \geqslant C\,\|X\|^2, \forall X \in \mathbb{R}^n \tag{1.2}$$

(here and hereafter, $||\cdot||$ will always denote the euclidean norm). Condition (1.2) is sufficient to ensure the well-posedness of (1.12), since the principal part $D\Delta$ in (1.12) is strongly parabolic in the sense of [19, Chapter 7, Definition 7].

More precisely, we consider the so-called null controllability problem, the goal being to drive the system to the null final target $Y(T) \equiv 0$ by a suitable choice of the controls $u = (u_1, \dots u_m) \in [L^2(\Omega)]^m$.

The scalar case (i.e. $n = 1$) has been analysed in [17] for a scalar potential $a \in \mathbf{H}$. Our goal here is to extend those results to coupled systems, obtaining a simple and exploitable spectral necessary and sufficient condition of controllability and the corresponding dual observability one.

The controllability and observability of systems of partial differential equations have been intensively studied in the last decade, leading to important progress. We shall refer to some of the existing literature in the end of this introduction. But, as indicated above, the number of articles devoted to non-local problems is very limited.

Our analysis will follow a combination of the methods developed in [22] for the analysis of parabolic systems and in [17] to handle non-local coupling terms. Accordingly, we shall use in an essential manner the spectral decomposition of the Laplacian.

Let $\{\lambda_k\}_{k\geq 1}$ be the eigenvalues of $-\Delta$ with Dirichlet boundary conditions and $e_k \in H_0^1(\Omega)$ be the corresponding eigenfunctions, constituting an orthonormal basis of $L^2(\Omega)$.

Before considering the non-locally perturbed case, let us first recall some recent results on models involving constant coefficient coupling terms:

$$\begin{cases} \partial_t Z = D^*\Delta Z + A^* Z & \text{in} \quad (0,T)\times\Omega, \\ Z(t,x) = 0 \quad \text{in} \quad (0,T)\times\partial\Omega, \\ Z(0) = Z^0, \end{cases}$$

where $Z^0 \in (L^2(\Omega))^n$ and $A^* \in \mathcal{M}_n(\mathbb{R})$ is a constant coupling matrix.

Here, rather than dealing with the controllability problem we consider the dual observability one. It concerns the obtention of full estimates on the state Z at time $t = T$ out of partial measurements on the control subsets ω_i.

In [22] it was proved that system (1.12) is observable on $(0,T)$ in the sense that there exists $C = C(T) > 0$ such that for every $Z^0 \in [L^2(\Omega)]^n$, the solution Z of (1.12) verifies

$$||Z(T)||^2 \leqslant C(T) \sum_{i=1}^{m} \int_0^T \int_{\omega_i} |B_i^* Z(t,x)|^2 dxdt \tag{1.3}$$

if and only if

$$\text{rank } K(\lambda_p) = n, \forall p \geqslant 1, \tag{1.4}$$

where

$$K(\lambda) := [B|(-\lambda D + A)B| \dots \ |(-\lambda D + A)^{n-1}B]. \tag{1.5}$$

Moreover, following [22, Proof of Theorem 3] and [24, Proof of Theorem 2.2], a precise upper bound on the observability constant $C(T)$ in (1.3) can be given for $T > 0$ small enough, getting:

$$||Z(T)||^2 \leqslant Ce^{\frac{C}{T}} \sum_{i=1}^{m} \int_0^T \int_{\omega_i} |B_i^* Z(t,x)|^2 dxdt. \tag{1.6}$$

If $A^* = 0$, it is easy to prove that (1.5) is equivalent to the following Kalman rank condition:

$$\text{rank } K_D = n, \tag{1.7}$$

where, by definition,

$$K_D := [B|DB| \dots |D^{n-1}B] \in \mathcal{M}_{n,nm}(\mathbb{R}), \tag{1.8}$$

that only concerns the coupling matrix D and the control one B. When $A^* \neq 0$ though, we get a sequence of spectral conditions, depending on the eigenvalues of the Laplacian.

In all what follows, we decompose the initial condition as

$$Z^0(x) = \sum_{k=1}^{\infty} Z_k e_k(x), \tag{1.9}$$

where $(Z_k)_{k\in\mathbb{N}^*} \in (l^2(\mathbb{N}^*))^n$.

The observability inequality (1.6), as pointed out in [18] (see also [24, Lemma 3.3] with $\beta = 1$ and $\alpha = 1/2$) can be rewritten, in terms of the Fourier series expansion of the initial datum Z^0 given in (1.9), as

$$\sum_{k=1}^{\infty} e^{-R\sqrt{\lambda_k}} ||Z_k||^2 \leqslant C(T) \sum_{i=1}^{m} \int_0^T \int_{\omega_i} |B_i^* Z(t,x)|^2 dxdt, \tag{1.10}$$

for some $R > 0$ and $C(T) > 0$ independent of $Z^0 \in [L^2(\Omega)]^n$.

Note that this kind of observability inequality (which is related to reachability issues, see e.g. [14]), introduced in [18, Remark 6.1], has also been used in [17, Lemma 2], for instance, to deal with non-local perturbations.

This spectral observability inequality motivates the introduction of the following Hilbert space of non-local potentials (that was mentioned before when describing the class of models under consideration)

$$\mathbf{H} := \left\{ f(x,\xi) = \sum_{m,j\geqslant 1} f_{mj} e_m(x) e_j(\xi) \in L^2(\Omega\times\Omega) | \sum_{m=1}^{\infty} \left(\sum_{j=1}^{\infty} \frac{1}{\lambda_j} |f_{mj}|^2 \right) \frac{1}{\lambda_m} e^{R\sqrt{\lambda_m}} < \infty \right\}, \tag{1.11}$$

$R > 0$ being as in (1.10).

Let us now consider the following (forward) adjoint system of (1.1) involving also the non-local coupling terms:

$$\begin{cases} \partial_t Z = D^*\Delta Z + \displaystyle\int_\Omega A^*(\xi,x)Z(t,\xi)d\xi \quad \text{in} \quad (0,T)\times\Omega, \\ Z(t,x)=0 \quad \text{in} \quad (0,T)\times\partial\Omega, \\ Z(0)=Z^0, \end{cases} \tag{1.12}$$

for some $Z^0 \in (L^2(\Omega))^n$.

Our goal is to extend the observability inequalities above for this complete model involving the non-local perturbations. We are able to reduce the observability problem under consideration to a unique continuation property for an elliptic problem, usually called Fattorini's Criterion [15]. This condition is much easier to be verified in practice, as illustrated by two examples in Section 3. As a consequence of the spectral observability inequality, by duality, we shall also derive the controllability property for the original control system involving the non-local terms.

The main result of this paper is the following.

Theorem 1. *Assume that $A(x,\xi)\in\mathcal{M}_n(\mathbf{H})$, where $\mathbf{H}$ is defined in (1.11), and that K_D verifies the Kalman rank condition (1.7).*

Then, there exists $C(T)>0$ such that any solution Z of (1.12) (involving the non-local perturbation terms) verifies

$$||Z(T)||^2 \leqslant C(T)\sum_{i=1}^m \int_0^T \int_{\omega_i} ||B_i^* Z(t,x)||^2 dxdt \tag{1.13}$$

if and only if the following unique continuation property is verified, for every $\lambda\in\mathbb{R}^n$:

$$\begin{cases} -D^*\Delta\varphi - \displaystyle\int_\Omega A^*(\xi,x)\varphi(\xi)d\xi = \lambda\varphi \ \text{in}\ \Omega, \\ \varphi = 0 \ \text{on}\ \partial\Omega, \\ B^*\varphi = 0 \ \text{in}\ \Omega, \\ \Rightarrow \varphi\equiv 0. \end{cases} \tag{1.14}$$

Equivalently, under condition (1.14), system (1.1) is null-controllable on $(0,T)$ (in the sense that for any $Y^0\in[L^2(\Omega)]^n$, there exists $u\in[L^2((0,T)\times\Omega)]^m$ such that the corresponding solution to (1.1) verifies $Y(T,\cdot)=0$)

The proof of the main result consists in obtaining the inequality (1.11) for the complete system (1.12) on the basis of the same inequality for the system in the absence of non-local perturbations (1.2). This is done applying a compactness-uniqueness argument, and reduces the issue to the fullfillement of the unique continuation property above (1.14) for the spectral problem. Once (1.11) is proved for the complete adjoint system (1.12), the null controllability result for (1.1) is a direct consequence of a classical duality principle.

Compactness-uniqueness arguments have rarely been applied in the context of heat equations because of the strong time irreversibility. In [17] this principle was applied in a satisfactory manner for scalar parabolic equations involving non-local potentials, provided they belong to the space $\mathbf{H}$. This compactness-uniqueness technique, which applies in the context of non-local perturbation terms, cannot be used for pointwise space-varying coupling terms. The main novelty of the present article is to extend this analysis to parabolic systems involving non-local terms.

Several other remarks are in order:

Remark 1.

- We are unable to derive an explicit estimate on the cost of controllability in small time, similar to the one given in (1.6), because we use a contradiction argument.
- Remark that in (1.14), $B^*\varphi = 0$ is assumed on all Ω and not only ω. This is a consequence of the analyticity properties of the kernel A^*. This fact facilitates the needed unique continuation property which is more of an algebraic nature since localisation (in the space variable) issues do not arise.

1.3 Bibliographical comments

As indicated above, there is an extensive literature devoted to the control of PDE systems but problems involving non-local terms are rarely considered. Some of the existing results concern the following topics and techniques (see also the survey [5] for earlier results). For a more detailed presentation, concerning also the hyperbolic and dispersive case, we refer to [22].

- One-dimensional results (i.e. $d = 1$) have been obtained in [6], [7], [4] and [9].
- Multi-dimensional results have been obtained in [12] for constant or time-dependent coupling terms, and partial results in the case of space-dependent coupling terms are obtained in [6], [20], [4], [8], [13] or [9].
- The nonlinear case has been studied in [16] and [11].
- To conclude, by means of the transmutation method from wave to heat systems (see [23] or [14]), one can deduce results for systems of heat-like equations out of the corresponding ones on wave-like models, under geometric conditions on the observation set of hyperbolic nature that are not natural in the parabolic setting. We refer to [3], [1] or [2].

2 Proof of the main result

Assuming that the spectral unique continuation property (1.14) is verified, the proof consists in showing that the null-controllability of (1.1) holds. To do this, using the equivalence between null controllability and observability, it suffices to show that the observability inequality (1.13) holds for the complete system (1.12).

The proof of this inequality for the complete system involving the non-local terms relies on a compactness-uniqueness argument similar to the one in [17, Proof of (16)]. We proceed in several steps.

Step 1: Splitting of the solution. To get (1.13), first of all, we decompose the solution Z of (1.12) into two parts $Z = \zeta + p$, where p verifies

$$\begin{cases} \partial_t p = D^* \Delta p & \text{in} \quad (0,T)\times\Omega, \\ p(0) = Z^0, \end{cases} \tag{2.1}$$

and ζ verifies

$$\begin{cases} \partial_t \zeta = D^* \Delta \zeta + \displaystyle\int_\Omega A^*(\xi,x)\zeta(t,\xi)d\xi + \int_\Omega A^*(\xi,x)p(t,\xi)d\xi & \text{in} \quad (0,T)\times\Omega, \\ \zeta(0) = 0. \end{cases} \tag{2.2}$$

From (1.7) and (1.10), we already know that

$$||p(T)||^2 \leqslant C \sum_{i=1}^m \int_0^T \int_{\omega_i} ||B_i^* p(t,x)||^2 dxdt. \tag{2.3}$$

Step 2: An auxiliary Carleman estimate.

Let us prove the following useful Carleman estimate.

Lemma 2.1. *There exist two constants $C_0 > 0$ (not depending on T) and $C(T) > 0$ such that for any $Z^0 \in [L^2(\Omega)]^n$, the solution p of (2.1) verifies*

$$\int_0^T \int_\Omega e^{-\frac{C_0}{t}} \left\|p(x,t)\right\|^2 dxdt \leqslant C(T) \sum_{k=1}^\infty \int_0^T \int_\Omega \sum_{i=1}^m ||B_i^* p(t,x) 1_{\omega_i}(x)||^2 dxdt. \tag{2.4}$$

Proof of Lemma 2.1. We follow the computations of [18, Remark 6.1]. First of all ,we decompose p in the Hilbert basis $\{e_k\}$ as

$$p(x) = \sum_{k=1}^{\infty} p_k e_k(x).$$

For $C_0 > 0$ (to be determined later on) ee remark that

$$\int_0^T \int_\Omega e^{-\frac{C_0}{t}} \|p(x,t)\|^2 \, dxdt = \sum_{k=1}^{\infty} \int_0^T e^{-\frac{C_0}{t}} ||\tilde{p}_k(t)||^2 dt, \tag{2.5}$$

where $\tilde{p}_k$ is the unique solution of the ordinary differential equation

$$\begin{cases} \tilde{p}_k'(t) = -\lambda_k D^* \tilde{p}_k(t) & \text{in} \quad (0,T)\times\Omega, \\ \tilde{p}_k(0) = p_k. \end{cases} \tag{2.6}$$

Using the ellipticity condition (1.2), there exists $C_1 > 0$ (independent of C_1) such that for any $t > 0$, one has

$$||\tilde{p}_k(t)||^2 \leqslant ||p_k||^2 e^{-C_1\lambda_k t}. \tag{2.7}$$

Hence, from (2.5) and (2.7) we deduce that

$$\int_0^T \int_\Omega e^{-\frac{C_0}{t}} \|p(x,t)\|^2 \, dxdt \leqslant \sum_{k=1}^{\infty} ||p_k||^2 \left(\int_0^T e^{-\frac{C_0}{t} - C_1\lambda_k t} dt \right). \tag{2.8}$$

Besides, it is well-known that, as $\lambda \to \infty$,

$$\int_0^T e^{-\frac{C_0}{t} - C_1\lambda_k t} dt \simeq \left(\frac{\pi^2 C_0}{C_1^3} \right)^{1/4} e^{-2\sqrt{C_1 C_1 \lambda}}.$$

Hence, there exists some $C_2 > 0$ such that for any $k > 0$, one has

$$\int_0^T e^{-\frac{C_0}{t} - C_1\lambda_k t} dt \leqslant C_2 e^{-2\sqrt{C_0 C_1 \lambda_k}}.$$

We deduce from (2.8) that

$$\int_0^T \int_\Omega e^{-\frac{C_0}{t}} \|p(x,t)\|^2 \, dxdt \leqslant C_2 \sum_{k=1}^{\infty} ||p_k||^2 e^{-2\sqrt{C_0 C_1 \lambda_k}}. \tag{2.9}$$

(2.4) follows by using (1.10) together with (2.9) and taking C_0 large enough such that $2\sqrt{C_0 C_1} > R$. ■

Step 3: Reduction to the proof of two inequalities.

We remark that in order to obtain (1.13), it is enough to prove the two following key inequalities:

$$\sum_{i=1}^{m} \int_0^T \int_{\omega_i} ||B_i^* p(t,x)||^2 dxdt \leqslant C \sum_{i=1}^{m} \int_0^T \int_{\omega_i} ||B_i^* Z(t,x)||^2 dxdt, \tag{2.10}$$

and

$$||Z(T)||^2 \leqslant C \sum_{i=1}^{m} \int_0^T \int_{\omega_i} ||B_i^* p(t,x)||^2 dxdt. \tag{2.11}$$

Step 4: Proof of (2.10).

Assume that (2.10) is not verified whereas (1.14) is verified. Then, there exists a sequence $(Z_n^0)_{n\in\mathbb{N}}$ such that the corresponding solution p_n of (2.1) with initial condition Z_n^0 verifies

$$\sum_{i=1}^{m}\int_0^T\int_{\omega_i}||B_i^*p(t,x)||^2dxdt = 1 \tag{2.12}$$

and the corresponding solution Z_n of (1.12) with initial condition Z_n^0 is such that

$$\sum_{i=1}^{m}\int_0^T\int_{\omega_i}||B_i^*Z_n(t,x)||^2dxdt < \frac{1}{n}. \tag{2.13}$$

We also call ζ_n the solution to (2.2) where p is replaced by p_n, so that we have the relation

$$Z_n = p_n + \zeta_n. \tag{2.14}$$

We are going to prove that $\zeta_n \to 0$ (up to a subsequence) strongly in $L^2((0,T)\times\Omega)$, which is obviously in contradiction with (2.12) and (2.13) since these estimates together with (2.14) imply

$$1 \leqslant 2\left(\frac{1}{n}+\sum_{i=1}^{m}\int_0^T\int_{\omega_i}||B_i^*\zeta_n(t,x)||^2dxdt\right).$$

First of all, let us remark that there exists $C>0$ such that

$$\left|\left|\int_\Omega A^*(\xi,x)Z_n(t,\xi)d\xi\right|\right|_{L^2((0,T),H^{-1}(0,L))} \leqslant C. \tag{2.15}$$

It is an easy consequence of the computation given in [17, (21)] applied on each component of A^*. Hence, by classical energy estimates and compactness arguments, one may assume that ζ_n converges strongly in $L^2((0,T)\times\Omega)$ to some $\zeta\in L^2((0,T)\times\Omega)$. This implies, together with (2.4) and (2.14), that if we fix $\delta\in(0,T)$, $(Z_n)_{n\in\mathbb{N}}$ is bounded in $L^2((\delta,T),\Omega)$. Hence, $(Z_n)_{n\in\mathbb{N}}$ can be assumed to converge weakly in $L^2((\delta,T),\Omega)$ to some $Z\in L^2((\delta,T),\Omega)$. Then, one can prove that Z solves the following PDE:

$$\partial_t Z = D^*\Delta Z + \int_\Omega A^*(\xi,x)Z(t,\xi)d\xi \quad \text{in} \quad (0,T)\times\Omega.$$

Moreover, we also know thanks to (2.13) that

$$B_i^*Z(t,x) = 0 \text{ on } (0,T)\times\omega_i,\ \forall i\in[|1,m|].$$

Using the well-known Fattorini criterion for approximate controllability [15], proving that $Z\equiv 0$ is equivalent to proving the following assertion:

$$\begin{cases} -D^*\Delta\varphi - \int_\Omega A^*(\xi,x)\varphi(\xi)d\xi = \lambda\varphi \text{ on } \Omega,\\ \varphi = 0 \text{ on } \partial\Omega,\\ B_i^*\varphi = 0 \text{ on } (0,T)\times\omega_i,\ \forall i\in[|1,m|],\\ \Rightarrow \varphi\equiv 0. \end{cases} \tag{2.16}$$

Let us consider any $\varphi\in L^2(\Omega)$ verifying

$$\begin{cases} -D^*\Delta\varphi - \int_\Omega A^*(\xi,x)\varphi(\xi)d\xi = \lambda\varphi \text{ on } \Omega,\\ \varphi = 0 \text{ on } \partial\Omega,\\ B_i^*\varphi(x) = 0 \text{ on } (0,T)\times\omega_i,\ \forall i\in[|1,m|]. \end{cases} \tag{2.17}$$

From (2.17) and an easy bootstrapping argument, we remark that $\varphi\in C^\infty(\Omega)$. We consider any a^* any component of A^* that we decompose as

$$a^*(\xi,x) = \sum_{m,j\geqslant 1} a^*_{mj}e_m(\xi)e_j(x),$$

Using condition (1.11) and since $A^* \in \mathcal{M}_n(\mathbf{H})$, we obtain that for any $j \in \mathbb{N}$, one has

$$\sum_{m=1}^{\infty} \frac{1}{\lambda_m} e^{R\sqrt{\lambda_m}} |a^*_{mj}|^2 < \infty,$$

implying thanks to (2.17) that for any $\varphi \in H^1_0(\Omega)^n$, one has $\mathcal{K}(\varphi) = 0$ on $\partial\Omega$, where $\mathcal{K}$ is given by

$$\mathcal{K} : \varphi \in L^2(\Omega) \mapsto \int_\Omega A^*(\xi, x)\varphi(\xi)d\xi.$$

Hence, another easy bootstrapping argument enables us to conclude that

$$\varphi \in \bigcap_{n=0}^{\infty} \mathcal{D}(\Delta^n)$$

where Δ represents here the Dirichlet Laplace operator with domain $H^1_0(\Omega) \cap H^2(\Omega)$). Let us now prove that φ is moreover analytic. Let $k \in \mathbb{N}$. In what follows, C is a constant that may vary from inequality to inequality and is independent of k. We consider the scalar product of the first line of (2.17) by the vector $\Delta^{2k+1}\varphi$ and we integrate on Ω. After some integrations by parts we obtain

$$||D^*\Delta^{k+1}\varphi||^2_{L^2(\Omega)} \leqslant \lambda||\nabla\Delta^k\varphi||^2_{L^2(\Omega)} + <\varphi, \Delta^{2k+1}(\mathcal{K}\varphi)>_{L^2(\Omega)}. \tag{2.18}$$

Using the definition of $\mathbf{H}$ given in (1.11), we remark that

$$||\mathcal{K}(\Delta^{2k+1}\varphi)||^2_{L^2(\Omega)} \leqslant C||\varphi||^2_{H^1_0(\Omega)}. \tag{2.19}$$

Using Young's inequality and plugging (2.19) into (2.18) we obtain that

$$||D^*\Delta^{k+1}\varphi||^2_{L^2(\Omega)} \leqslant C\lambda||\varphi||^2_{H^{2k+1}} + ||\varphi||^2_{L^2(\Omega)} + C||\varphi||^2_{H^1_0(\Omega)} \leqslant C\left(\lambda||\varphi||^2_{H^{2k+1}} + ||\varphi||^2_{H^1_0(\Omega)}\right). \tag{2.20}$$

Using the ellipticity condition (1.2), we obtain from (2.20) that

$$||\varphi||^2_{H^{2k+2}} \leqslant C\left(\lambda||\varphi||^2_{H^{2k+1}} + ||\varphi||^2_{H^1_0(\Omega)}\right). \tag{2.21}$$

Easy interpolation arguments together with an induction enable us to obtain from (2.21) that for any $m \in \mathbb{N}^*$, one has

$$||\varphi||^2_{H^m} \leqslant C\left(1 + \lambda + \cdots + \lambda^{m-1}\right)||\varphi||^2_{H^1_0(\Omega)},$$

which implies the analyticity of φ.

We deduce that $B^*\varphi$ is also analytic on Ω, hence, using the last line of (2.17), we deduce that

$$B^*\varphi = 0 \text{ in } \Omega.$$

Now, using assumption (1.14), (2.16) is verified and hence $Z \equiv 0$ on $(0,T) \times \Omega$. We deduce that p_n converges weakly to $-\zeta$ in $L^2((0,T) \times \Omega)$, which implies that $\zeta_n \to 0 = \zeta$ in $L^2((0,T) \times \Omega)$ because of (2.2). This leads to the desired contradiction. ■

Step 5: Proof of (2.11).

This inequality is a consequence of (2.3) and easy energy estimates on ξ using equation (2.2) and arguing as in the proof of [17, (21)].

■

Finally, we have proved that (1.14) implies (1.13). The fact that the null-controllability of (1.1) (i.e. (1.13)) implies (1.14) is standard and is omitted. ■

3 Two simple examples of application

3.1 Indirect controllability of cascade systems of two equations

In what follows, we consider the case of two coupled equations with cascade structure and control on the first component.

More precisely, we consider the following system:

$$\begin{cases} \partial_t Y^1 = d_{11}\Delta Y^1 + d_{12}\Delta Y^2 + \displaystyle\int_\Omega a_{11}(x,\xi)Y^1(t,\xi)d\xi + \int_\Omega a_{12}(x,\xi)Y^2(t,\xi)d\xi + u1_\omega & \text{in} \quad (0,T)\times\Omega, \\ \partial_t Y^2 = d_{21}\Delta Y^1 + d_{22}\Delta Y^2 + \displaystyle\int_\Omega a_{21}(x,\xi)Y^1(t,\xi)d\xi & \text{in} \quad (0,T)\times\Omega, \\ Y^1(t,x) = Y^2(t,x) = 0 & \text{in} \quad (0,T)\times\partial\Omega, \\ (Y^1(0),Y^2(0)) = (Y^1_0,Y^2_0) & \text{in} \quad \Omega. \end{cases} \tag{3.1}$$

Here D is given by

$$D := \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

and is assumed to verify (1.2). The non-local potential A is given by

$$A(x,\xi) := \begin{pmatrix} a_{11}(x,\xi) & a_{12}(x,\xi) \\ a_{21}(x,\xi) & 0 \end{pmatrix},$$

where $a_{ij}(x) \in \mathbf{H}$ for $i,j = 1,2$. We consider the control operator B given by

$$B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The control acts on some open subset $\omega \subset \Omega$. We are going to prove the following sufficient condition for the controllability of (3.1).

Theorem 2. *Assume that $d_{21} \neq 0$ and $d_{22} \neq 0$. Then, (3.1) is null-controllable.*

Proof of Theorem 2. First, observe that condition (1.7) is equivalent to $d_{21} \neq 0$. Then, applying Theorem 1, the null-controllability of (3.1) is equivalent to the following unique continuation property:

$$\begin{cases} -d_{21}\Delta\varphi - \displaystyle\int_\Omega a_{21}(\xi,x)\varphi(\xi)d\xi = 0 & \text{in } \times\Omega, \\ -d_{22}\Delta\varphi = \lambda\varphi & \text{in } \times\Omega, \\ \varphi = 0 & \text{in } \partial\Omega, \\ \Rightarrow \varphi \equiv 0. \end{cases} \tag{3.2}$$

By contradiction, assume that there exists some $\varphi \not\equiv 0$ verifying the three first equations of (3.2). Let us decompose a_{21} as follows:

$$a_{21}(\xi,x) := \sum_{k,l} c_{kl}e_k(x)e_l(\xi).$$

Since $d_{22} \neq 0$, it is clear from the second equation of (3.2) that there exists $m \in \mathbb{N}^*$ such that $\lambda = d_{22}\lambda_m$. In this case, without loss of generality we may assume that $\varphi(x) = e_m(x)$. Using the spectral decomposition of a_{21}, we obtain that

$$\int_\Omega a_{21}(\xi,x)\varphi(\xi)d\xi = \sum_k c_{km}e_k(x).$$

Moreover, one has $-d_{21}\Delta\varphi(x) = d_{21}\lambda_m e_m(x)$. Hence, we deduce that a_{21} is necessarily such that the two following conditions are verified:

- $c_{km} = 0$ if $k \neq m$.
- $c_{mm} = d_{21}\lambda_m$.

The conclusion follows since such an a_{21} cannot be in $\mathbf{H}$ in view of (1.11). ■

3.2 Simultaneous controllability of two equations with diagonal principal part

In what follows, we consider the case of two coupled equations with simultaneous control:

$$\left\{\begin{array}{lll} \partial_t Y^1 = d_{11}\Delta Y^1 + \displaystyle\int_\Omega a_{11}(x,\xi)Y^1(t,\xi)d\xi + \int_\Omega a_{12}(x,\xi)Y^2(t,\xi)d\xi + 1_\omega u & \text{in} & (0,T)\times\Omega, \\ \partial_t Y^2 = d_{22}\Delta Y^2 + \displaystyle\int_\Omega a_{21}(x,\xi)Y^1(t,\xi)d\xi + \int_\Omega a_{22}(x,\xi)Y^2(t,\xi)d\xi + 1_\omega u & \text{in} & (0,T)\times\Omega, \\ Y^1(t,x) = Y^2(t,x) = 0 & \text{in} & (0,T)\times\partial\Omega, \\ (Y^1(0),Y^2(0)) = (Y^1_0,Y^2_0) & \text{in} & \Omega. \end{array}\right. \tag{3.3}$$

Here D is given by

$$D := \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}$$

where $d_{11} > 0$ and $d_{22} > 0$. A is given by

$$A(x,\xi) := \begin{pmatrix} a_{11}(x,\xi) & a_{12}(x,\xi) \\ a_{21}(x,\xi) & a_{22}(x,\xi) \end{pmatrix},$$

where $a_{ij}(x) \in \mathbf{H}$ for $i,j = 1,2$. We consider the control operator B given by

$$B := \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The control acts on some open subset $\omega \subset \Omega$.

We are going to prove the following sufficient condition for the controllability of (3.3).

Theorem 3. *Assume that $d_{11} \neq d_{22}$. Then, (3.3) is null-controllable if the following conditions (for instance) are verified:*

- $a_{11} = a_{21}$.
- *a_{12} and a_{22} are symmetric in the variables (x,ξ).*

Proof of Theorem 3. Remark that the Kalman rank condition (1.7) is verified here since $d_{11} \neq d_{22}$ and each component of B is nonzero. Hence, we can apply Theorem 1 and we obtain that the null-controllability of (3.3) is equivalent to the following unique continuation property:

$$\left\{\begin{array}{lll} -d_{11}\Delta\varphi^1 - \displaystyle\int_\Omega a_{11}(\xi,x)\varphi^1(\xi)d\xi - \int_\Omega a_{21}(\xi,x)\varphi^2(\xi)d\xi = \lambda\varphi^1 & \text{in} & \Omega, \\ -d_{22}\Delta\varphi^2 - \displaystyle\int_\Omega a_{12}(\xi,x)\varphi^1(\xi)d\xi - \int_\Omega a_{22}(\xi,x)\varphi^2(\xi)d\xi = \lambda\varphi^2 & \text{in} & (0,T)\times\Omega, \\ \varphi^1 + \varphi^2 = 0 & \text{in } \Omega, & \\ \varphi^1 = \varphi^2 = 0 & \text{on } \partial\Omega, & \\ \Rightarrow \varphi^1 = \varphi^2 = 0 & \text{in } \Omega. & \end{array}\right. \tag{3.4}$$

Substituting φ^2 in the first two equations of (3.4) and using the hypothesis $a_{11} = a_{21}$, we obtain that (3.4) is equivalent to

$$\left\{\begin{array}{lll} -d_{11}\Delta\varphi^1 = \lambda\varphi^1 & \text{in} & \Omega, \\ -d_{22}\Delta\varphi^1 - \displaystyle\int_\Omega a_{12}(\xi,x)\varphi^1(\xi)d\xi + \int_\Omega a_{22}(\xi,x)\varphi^1(\xi)d\xi = \lambda\varphi^1 & \text{in} & (0,T)\times\Omega, \\ \varphi^1(x) = 0 \quad \text{on } \partial\Omega, & & \\ \Rightarrow \varphi^1 = 0 \text{ in } \Omega. & & \end{array}\right. \tag{3.5}$$

From the first line of (3.5) we may assume that $\lambda > 0$ (since every eigenvalue of the Laplace operator with Dirichlet boundary conditions is positive). We multiply the first line of (3.5) by d_{22} and the second line of (3.5) by d_{11}, and we subtract the result. We obtain that

$$d_{11}\int_\Omega a_{12}(\xi,x)\varphi^1(\xi)d\xi - d_{11}\int_\Omega a_{22}(\xi,x)\varphi^1(t,\xi)d\xi = \lambda(d_{22}-d_{11})\varphi^1. \tag{3.6}$$

We apply the Laplace operator to this equation, we use the symmetry of the coefficients $a_{12},\ a_{22}$ and we perform some integrations by parts. We obtain that

$$d_{11}\int_\Omega a_{12}(\xi,x)\Delta\varphi^1(\xi)d\xi - d_{11}\int_\Omega a_{22}(\xi,x)\varphi^1(t,\xi)d\xi = \lambda(d_{22}-d_{11})\Delta\varphi^1.$$

Now, we replace $\Delta\varphi^1$ thanks to the first line of (3.5) and we obtain

$$-\lambda d_{11}\int_\Omega a_{12}(\xi,x)\varphi^1(\xi)d\xi + \lambda d_{11}\int_\Omega a_{22}(\xi,x)\varphi^1(t,\xi)d\xi = \lambda^2(d_{22}-d_{11})\varphi^1. \tag{3.7}$$

Multiplying (3.6) by λ and using (3.7) leads to $\varphi^1 = 0$ since $\lambda \neq 0$ and $d_{11} \neq d_{22}$, so that we also have $\varphi^2 = 0$ by the third line of (3.4).

■

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