UIMP, Santander, July 04

Lack of collision for a simplied 1-d model for fluid-solid interaction

Enrique Zuazua
Universidad Aut
ponoma
28049 Madrid, Spain
enrique.zuazua@uam.es

J. VAZQUEZ & E. Z:, 2002, 2004

We consider a simplified model for a 1-D fluid containing a finite number N of point particles floating in it and located at the points $x = h_i(t)$, $i = 1, \cdots, N$ moving with the fluid. Without loss of generality we may assume that, in the absence of collision,

$$h_1(t) < h_2(t) < \cdots < h_N(t), \tag{1}$$

holds for some time $0 < t < T$. The complete system under consideration reads as follows:

$$\begin{cases} u_t - u_{xx} + \kappa(u^2)_x = 0, & x \in I_i(t), \quad i = 0, \cdots, N, \quad t > 0 \\ h_i'(t) = u(h_i(t), t), & i = 1, \cdots, N, \quad t > 0 \\ m_i h_i''(t) = [u_x](h_i(t), t), & i = 1, \cdots, N, \quad t > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \\ h_i(0) = h_{i,0}, & h_i'(0) = h_{i,1}, \quad i = 1, \cdots, N. \end{cases} \tag{2}$$

Here and in the sequel, $I_i(t)$ stand for the intervals occupied by the fluid, separated by the point particles. The masses m_i, $i = 1, \cdots, N$, of the particles are positive real numbers. The coefficient κ measures the ratio between convection and diffusion, which for the heat equation it is zero.

$[f](x_0)$ denotes the jump of a space function $f(x)$ at the point x_0,

$$[f](x) = \lim_{s \to 0} (f(x+s) - f(x-s)).$$

The velocities of the fluid and the particles coincide, and each particle is accelerated by the difference of the velocity gradient on both sides of it. Thus, the velocity gradient acts as a pressure.

The system can be viewed as a free-boundary problem. One may construct local in time solutions with the regularity that corresponds to the usual strong solutions of fluid flow theory. That is to say: $\in C([0,T];H^1)\cap L^2(0,T;H^2))$.

MAIN RESULT: the particle trajectories of these solutions never meet in finite time, so that the strong solution can be uniquely continued for all time.

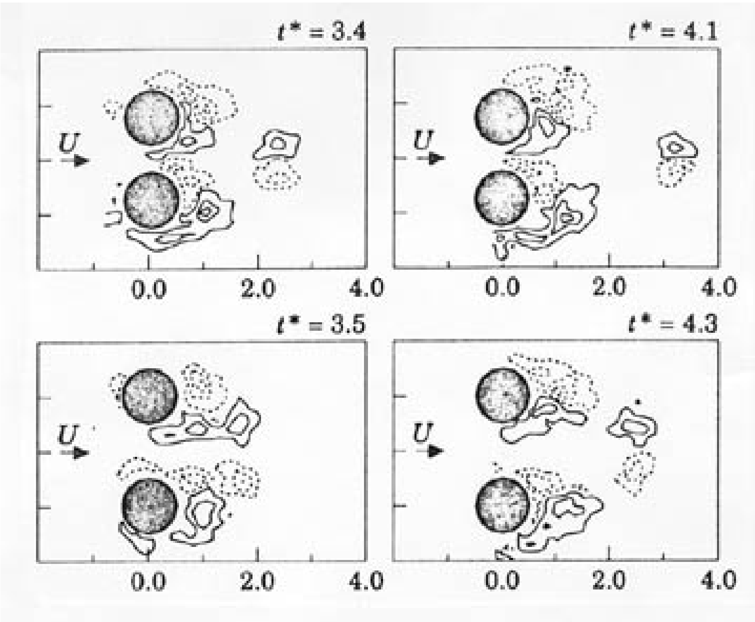
The occurs for the Dirichlet problem in a bounded interval or on the half line, whose boundary represents a rigid wall. In that case, we prove that no finite-time collision with the wall occurs and the strong solution exists globally in time.

Once no-collision is proved, solutions are global in time: We can the also obtain their asymptotic form as $t \to \infty$:

* The fluid solution converges to a self-similar solution of the Burgers equation. Thus, to a first order, the effct of the masses vanishes.

* Then, the particles, float freely in the fluid, following parabolic trajectories that we can explicitly compute.

The problem of collision for fluids governed by the Navier-Stokes equations is open in both variants: whether two rigid bodies may collide or whether one rigid body may collide with the exterior boundary.



Theorem 1 (Existence of strong solutions without collision) *Let $u_0 \in H^1(\mathbb{R})$, and let us for $i = 1, \cdots, N$ have quantities $h_{i,0}$, $h_{i,1}, m_i \in \mathbb{R}$ be such that*

$$-\infty < h_{1,0} < h_{2,0} < \cdots < h_{N,0} < \infty, \tag{3}$$

and $m_i > 0$. Under the compatibility conditions

$$u_0(h_{i,0}) = h_{i,1}, \quad i = 1, \cdots, N, \tag{4}$$

there exists a unique global solution $(u, h_1(t), \cdots, h_N(t))$ of Problem (**P**) *such that*

$$u \in C([0, \infty); H^1(\mathbb{R})), \quad u_{xx} \in L^2(Q^i), \, i = 0, ..., N, \tag{5}$$

where $Q^i = \{(x,t) : x \in I_i(t), 0 < t < T\}$, while the h_i's satisfy

$$h_i(t) \in C^1([0, \infty); \mathbb{R}), \, i = 1, \cdots, N, \quad h_i'' \in L^2(0, T), \tag{6}$$

for all finite $T > 0$*, and the initial data are taken. Moreover, the particles do not collide in finite time,*

$$h_1(t) < h_2(t) < \cdots < h_N(t), \quad \forall t > 0. \tag{7}$$

Theorem 2 (Asymptotic behavior) *Under the assumptions of Theorem 1 and assuming further that $u_0 \in L^1(\mathbb{R})$ it follows that*

$$t^{(1-1/p)/2} \parallel u(t) - U(t) \parallel_{L^p} \to 0, \text{ as } t \to \infty \tag{8}$$

for all $1 \leqslant p \leqslant \infty$, where

$$U(x,t) = \frac{1}{t^{1/2}} f_M(x/\sqrt{t}) \tag{9}$$

is the self-similar solution of Burger's equation, satisfying

$$\begin{cases} u_t - u_{xx} + \kappa (u^2)_x = 0, & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = M\delta_0, \end{cases} \tag{10}$$

where δ_0 denotes the Dirac delta at the origin and the asymptotic momentum M is given by

$$M = \int_{\mathbb{R}} u_0(x) dx + \sum_{i=1}^{N} m_i h_{i,1}. \tag{11}$$

If $M > 0$, we also have

$$t^{-1/2}\Big|h_i(t) - c\sqrt{t}\Big| \to 0, \text{ as } t \to \infty, i = 1, \cdots, N \tag{12}$$

where $c > 0$ is uniquely determined by the equation

$$f_M(c) = c/2. \tag{13}$$

Moreover,

$$t^{1/2}\Big|h_i'(t) - \frac{c}{2\sqrt{t}}\Big| \to 0, t \to \infty, i = 1, \cdots, N. \tag{14}$$

STRATEGY:

Assuming that the point particles are located at different points at the initial time (which is one of the main assumptions of Theorem 1 above), a change of variables allows to reduce it to finding a fixed point in a suitable functional setting. Methods of the theory of evolution equations and fixed point arguments allow us to prove the *local in time* existence and uniqueness. Then, solutions may be extended to the maximal existence time $[0, T)$. Global existence is then equivalent to the fact that $T = \infty$. The study of global solvability needs uniform global estimates.

The *energy*

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} u^2 dx + \frac{1}{2}\sum_{i=1}^{N} m_i \left|h_i'(t)\right|^2 \tag{15}$$

is dissipated along trajectories according to the *dissipation law*

$$\frac{dE(t)}{dt} = - \int_{\mathbb{R}} |u_x(x,t)|^2 dx. \tag{16}$$

This shows that the energy does not blow-up while solutions exist.

The existence of strong solutions needs additional estimates which are obtained essentially by using u_{xx} as test function.

However, these facts are not sufficient to guarantee *global existence*. Indeed, the classical continuation argument allowing to extend the solution for all $t > 0$ requires of knowing a priori that the particles do not collide. Thus, the key point in the proof of Theorem 1 is establishing the lack of collision property (7). Our proof of that fact is based on the simple observation that the particle dynamics is governed by the ordinary differential equation (ODE):

$$h_i'(t) = u(h_i(t), t), t > 0, \quad i = 1, \cdots, N. \tag{17}$$

Thus, if two particles collide in, say, time $t = T > 0$, (without loss of generality we may assume that the colliding particles are located at the points $h_1(t)$ and $h_2(t)$), both solve the Cauchy problem

$$\begin{cases} h_i'(t) = u(h_i(t), t), & 0 \leqslant t < T, i = 1, 2 \\ h_1(T) = h_2(T). \end{cases} \tag{18}$$

Problem: to show uniqueness of solutions backwards in time. It is sufficient to show that $u(h, t)$ is Lipschitz continuous in the space variable h, since u enters in (18) as the nonlinearity of the ODE, together with some integrability property in time. This regularity is proved by means of a local H_x^2-estimate, which is a crucial step in our result.

It is basically the same estimate we needed before to construct the strong solutions but with the key observation that there is a version which is valid up to the possible collision time with bounds independent of the distance between consecutive particles. Collision is then eliminated as a consequence of the obtained regularity.