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The heat equation

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Plan:

3.- The heat equation:

3.1 Preliminaries on the control of the heat equation.

3.2. Hyperbolic control implies parabolic one

3.3 Numerical approximation in 1-d

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3.5 A first result in the multi-dimensional case

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Preliminaries on the control of the heat equation.

THE GENERAL PROBLEM: NULL CONTROL OR CONTROL TO ZERO

TO CONTROL TO THE NULL EQUILIBRIUM STATE PARABOLIC EQUATIONS BY MEANS OF A CONTROL (RIGHT HAND SIDE TERM) CONCENTRATED ON AN OPEN SUBSET OF THE DOMAIN WHERE THE EQUATION HOLDS.

EQUIVALENT FORMULATION: OBSERVABILITY

ANALYZE HOW MUCH OF THE TOTAL ENERGY OF SOLUTIONS CAN BE OBTAINED OUT OF LOCAL MEASUREMENTS.

OBSERVATION $\equiv$ CONTROL

THE CONTROL PROBLEM

Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

1_ω denotes the characteristic function of the subset ω of Ω where the control is active.

We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (1) admits an unique solution

$$u \in C\left([0,T]\,; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right).$$

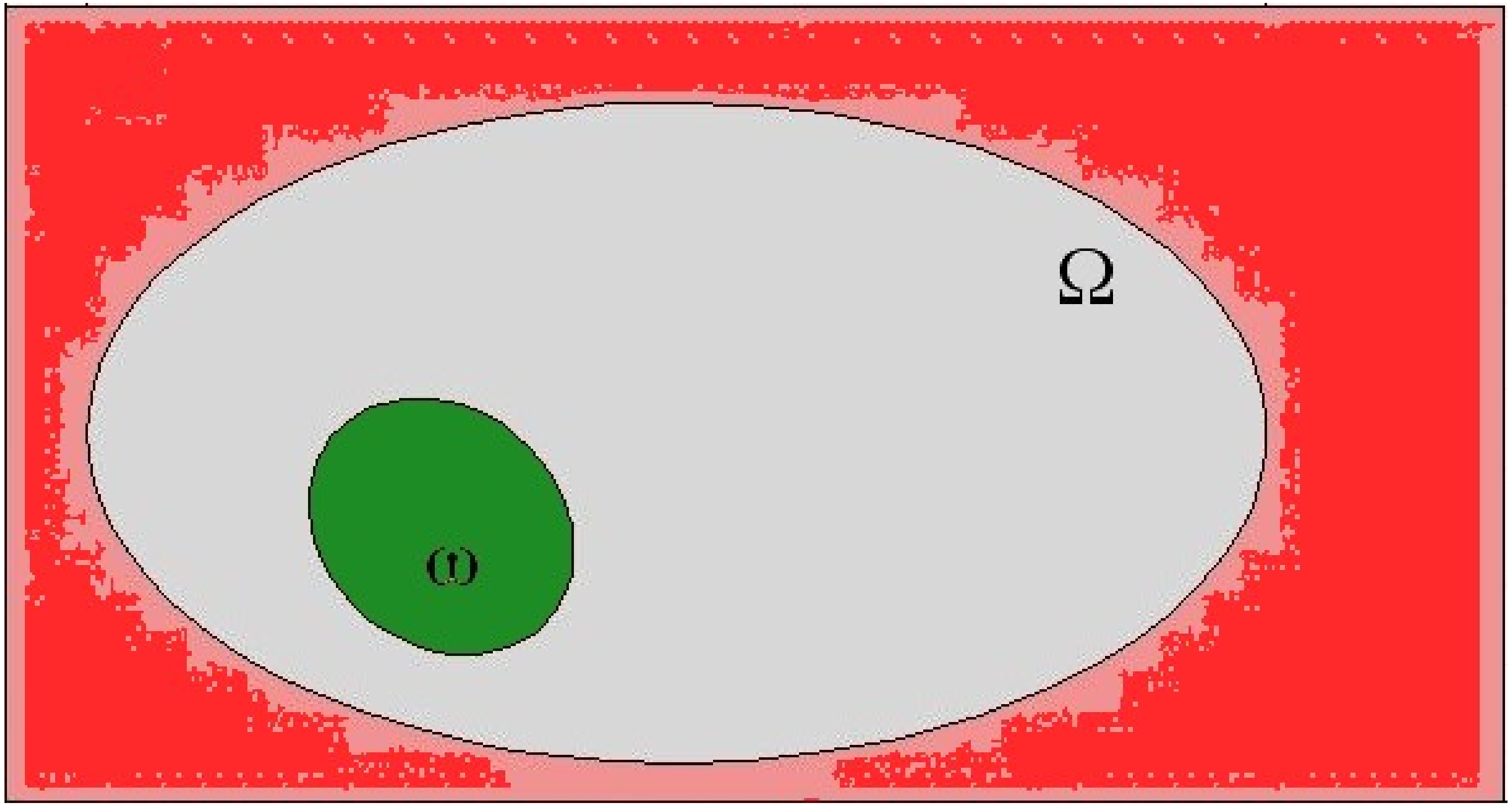
$$u = u(x,t) = solution = state,\ f = f(x,t) = control$$

Goal: To produce prescribed deformations on the solution u by means of suitable choices of the control function f.

We introduce the reachable set $R(T; u^0) = \left\{ u(T) : f \in L^2(Q) \right\}$.

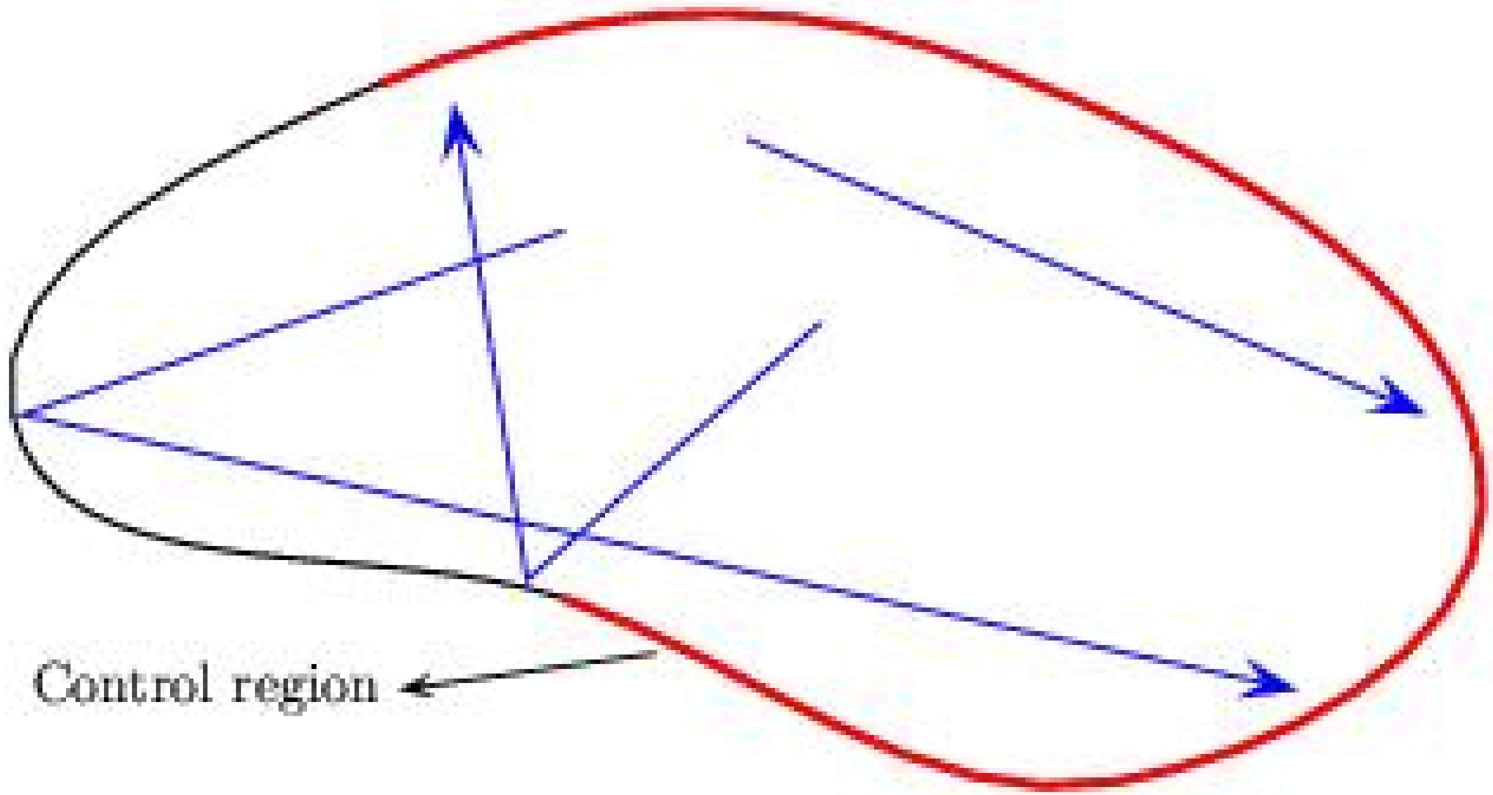
Approximate controllability: $R(T; u^0)$ is dense in $L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.

Null controllability: if $0 \in R(T; u^0)$ for all $u^0 \in L^2(\Omega)$.


Ω
ω

In principle, due to the intrinsic infinite velocity of propagation of the heat equation, one can not exclude these properties to hold in any time $T > 0$ and from any open non-empty open subset ω of Ω.

Note that for similar properties to hold for wave equations typically one needs to impose geometric conditions on the control subset and the time of control, namely, the so called GCC (Geometric Control Condition) by Bardos-Lebeau-Rauch: It asserts, roughly, that all rays of geometric optics enter the control set ω in time T.



But this kind of Geometric Condition in unnecessary for the heat equation.

Approximate controllability

For all initial data u^0, all final data $u^1 \in L^2(\Omega)$ and all $\varepsilon > 0$ there exists a control f_ε such that the solution satisfies:

$$||u(T) - u^1||_{L^2(\Omega)} \leq \varepsilon.$$

- Approximate controllability does not guarantee that the target u^1 may be reached exactly. It could well be that $||f_\varepsilon||_{L^2(\Omega)} \to \infty$ as $\varepsilon \to 0$.

- The property as such is of little use in practice since too large controls might be impossible to implement.

Approximate controllability is in fact equivalent to an unique continuation property for the adjoint:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{2}$$

More precisely, approximate controllability holds if and only if the following uniqueness or unique continuation property (UCP) is true:

$$\varphi = 0 \text{ in } \omega \times (0,T) \implies \varphi \equiv 0, \text{ i.e. } \varphi^0 \equiv 0. \tag{3}$$

This UCP is a consequence of Holmgren's uniqueness Theorem.

This is so for all ω and all $T > 0$.

UCP $\Longrightarrow$ Approximate controllability*

Consider the functional

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T\int_\omega \varphi^2 dx dt + \varepsilon \left|\varphi^0\right|_{L^2(\Omega)} - \int_\Omega \varphi^0 u^1 dx + \int_\Omega \varphi(0) u^0 dx. \quad (4)$$

$J_\varepsilon : L^2(\Omega) \to \mathbb{R}$ is continuous, and convex.

Moreover, UCP implies coercivity:

$$\lim_{||\varphi^0||_{L^2(\Omega)}\to\infty} \frac{J_\varepsilon(\varphi^0)}{||\varphi^0||_{L^2(\Omega)}} \geq \varepsilon.$$

*C. Fabre, J. P. Puel & E. Z. Approximate controllability for the semilinear heat equation. Proc. Roy. Soc. Edinburgh, 125A (1995), 31-61.

Accordingly, the minimizer $\widehat{\varphi}^0$ exists and the control

$$f_\varepsilon = \widehat{\varphi}$$

where $\widehat{\varphi}$ is the solution of the adjoint system corresponding to the minimizer is the control such that

$$||u(T) - u^1||_{L^2(\Omega)} \leq \varepsilon.$$

This is a general principle:†

UCP $\Longrightarrow$ APPROXIMATE CONTROLLABILITY

†In fact one can prove that (UCP) implies a stronger result, namely, that, together with the ε-distance property, one can also show that the projection over a finite-dimensional subspace E can be reached exactly. This can be done by minimizing the functional $J_{E,\varepsilon}(\varphi^0) = \frac{1}{2}\int_0^T\int_\omega \varphi^2 dxdt + \varepsilon \left|(I - \pi_E)\varphi^0\right|_{L^2(\Omega)} - \int_\Omega \varphi^0 u^1 dx + \int_\Omega \varphi(0) u^0 dx.$

This argument does not provide any estimate on the size of the control f_ε as $\varepsilon \to 0$. Roughly speaking:

- For a very narrow set of exactly reachable u^1 states the controls f_ε are bounded and converge as $\varepsilon \to 0$ to a control f such that

$$u(T) = u^1.$$

  This necessarily happens for a small class of u^1 because of the regularizing effect of the heat equation.

- Typically, for targets u^1 which are in a Sobolev class, the controls f_ε diverge exponentially on $1/\varepsilon^\alpha$, for some α depending on the Sobolev class they belong to.

Null controllability

For achieving $u(T) = 0$ we have to consider the case in which

$$u^1 = 0,\ \varepsilon = 0.$$

Thus, we are led to considering the functional

$$J_0(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0)u^0 dx \tag{5}$$

Obviously, the functional is continuous and convex from $L^2(\Omega)$ to $\mathbb{R}$.

Is it coercive?

For coercivity the following observability inequality is needed:

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{6}$$

This inequality is very likely to hold: because of the very strong regularizing effect of the heat equation the norm of $\varphi(0)$ is a very weak measure of the total size of solutions. Indeed, in a Fourier series representation, the norm of $\varphi(0)$ presents weights which are of the order of $\exp(-\lambda_j T)$, $\lambda_j \to \infty$ being the eigenvalues of the Dirichlet $-\Delta$.

For the wave equation this inequality requires of the GCC (sufficiently large time and geometric conditions on the subset ω to absorbe all rays of Geometric Optics). But for the heat equation there is no reason to think on the need of any restriction on T or ω.

Actually, this estimate was proved By Fursikov and Imanuvilov (1996)[‡] using Carleman inequalities. In fact the same proof applies for equations with smooth (C^1) variable coefficients in the principal part and for heat equations with lower order potentials.

[‡]A. V. Fursikov and O. Yu. Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series # 34, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, 1996

Consider the heat equation or system with a potential $a = a(t,x)$ in $L^\infty(Q;\mathbb{R}^{N\times N})$:

$$\begin{cases} \varphi_t - \Delta\varphi + a\varphi = 0, & \text{in } Q, \\ \varphi = 0, & \text{on } \Sigma, \\ \varphi(0,x) = \varphi^0(x), & \text{in } \Omega, \end{cases} \tag{7}$$

where φ takes values in $\mathbb{R}^N$.

Note that we have reversed the sense of time to make the inequality more intuitive and better underline the effect of the heat equation as time evolves: regularizing effect and possible exponential increase on the size of the solution due to the presense of the potential as Gronwall's inequality predicts.

Theorem A (Fursikov+Imanuvilov, 1996, E. Fernández-Cara+E. Zuazua, 2000) §

Assume that ω is an open non-empty subset of Ω. Then, there exists a constant $C = C(\Omega, \omega) > 0$, depending on Ω and ω but independent of T, the potential $a = a(t, x)$ and the solution φ of (7), such that

$$\| \varphi(T) \|^2_{(L^2(\Omega))^N} \leq \exp\left(C\left(1 + \frac{1}{T} + T \| a \|_\infty + \| a \|_\infty^{2/3}\right)\right) \int_0^T \int_\omega |\varphi|^2 dx dt, \tag{8}$$

for every solution φ of (7), potential $a \in L^\infty(Q; \mathbb{R}^{N \times N})$ and time $T > 0$.

§ E. Fernández-Cara& E. Z. The cost of approximate controllability for heat equations: The linear case. Advances in Differential Equations, 5 (4-6) (2000), 465–514.

Sketch of the proof:

Introduce a function $\eta^0 = \eta^0(x)$ such that:

$$\begin{cases} \eta^0 \in C^2(\bar{\Omega}) & \\ \eta^0 > 0 & \text{in} \quad \Omega,\ \eta^0 = 0 \quad \text{in} \quad \partial\Omega \\ \nabla\eta^0 \neq 0 & \text{in} \quad \overline{\Omega\backslash\omega}. \end{cases} \tag{9}$$

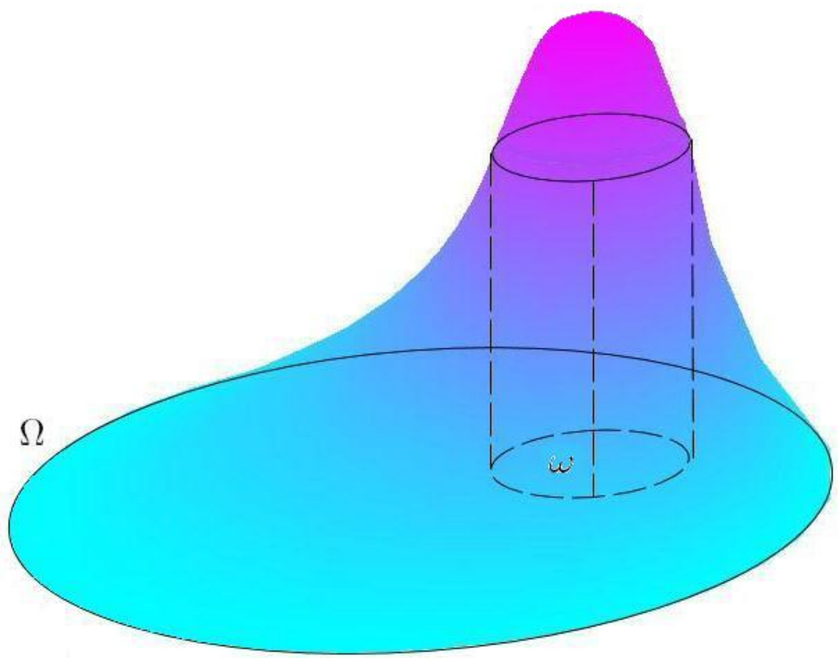
In some particular cases, for instance when Ω is star-shaped with respect to a point in ω, it can be built explicitly without difficulty. But the existence of this function is less obvious in general, when the domain has holes or its boundary oscillates, for instance.

Let $k > 0$ such that $k \geq 5\max_{\bar{\Omega}} \eta^0 - 6\min_{\bar{\Omega}} \eta^0$ and let

$$\beta^0 = \eta^0 + k, \bar{\beta} = \frac{5}{4}\max \beta^0,\ \rho^1(x) = e^{\lambda\bar{\beta}} - e^{\lambda\beta^0}$$

with $\lambda, \bar{\beta}$ sufficiently large. Let be finally

$$\gamma = \rho^1(x)/(t(T-t)); \rho(x,t) = \exp(\gamma(x,t)).$$



The following Carleman inequality holds:

Proposition 1 *(Fursikov + Imanuvilov, 1996)*

There exist positive constants $C_*, s_1 > 0$ *such that*

$$\frac{1}{s}\int_Q \rho^{-2s} t(T-t)\left[|q_t|^2 + |\Delta q|^2\right] dxdt \tag{10}$$
$$+s\int_Q \rho^{-2s} t^{-1}(T-t)^{-1}|\nabla q|^2\, dxdt + s^3\int_Q \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt$$
$$\leq C_*\left[\int_Q \rho^{-2s}|\partial_t q - \Delta q|^2\, dxdt + s^3\int_0^T\int_\omega \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt\right]$$

for all $q \in Z$ *and* $s \geq s_1$.

Moreover, C_* *depends only on* Ω *and* ω *and* s_1 *is of the form*

$$s_1 = s_0(\Omega, \omega)(T + T^2).$$

Let us go back to the estimate:

$$\| \varphi(T) \|^2_{(L^2(\Omega))^N} \leq \exp\left(C\left(1+\frac{1}{T}+T \| a \|_\infty + \| a \|_\infty^{2/3}\right)\right) \int_0^T \int_\omega |\varphi|^2 dx dt. \tag{11}$$

Three different terms have to be distinguished on the observability constant on the right hand side:

$$C_1^*(T, a) = \exp\left(C\left(1+\frac{1}{T}\right)\right), \quad C_2^*(T, a) = \exp(CT \| a \|_\infty), \tag{12}$$

$$C_3^*(T, a) = \exp\left(C \| a \|_\infty^{2/3}\right).$$

The role of the first two constants is clear: The first one $C_1^*(T, a) = \exp\left(C\left(1+\frac{1}{T}\right)\right)$ takes into account the increasing cost of making continuous observations as T diminishes. The second one $C_2^*(T, a) =$

$\exp(CT \parallel a \parallel_\infty)$ is due to the use of Gronwall's inequality to pass from a global estimate in (x,t) into an estimate for $t = T$.

What about the third one?

Theorem 1 [¶]

The third constant $C_3^*(T,a)$ is sharp in the range

$$\parallel a \parallel_\infty^{-2/3} \lesssim T \lesssim \parallel a \parallel_\infty^{-1/3}, \tag{13}$$

for systems $N \geq 2$ and in more than one dimension $n \geq 1$.

Open problem: Optimality for scalar equations $(N = 1)$ and in one space dimension $(n = 1)$.

[¶] Th. Duyckaerts, X. Zhang and E. Z., Annales IHP, 2006, to appear.

The proof is based on the following Theorem by V. Z. Meshkov, 1991.

Theorem 2 (Meshkov, 1991). *Assume that the space dimension is* $n = 2$. *Then, there exists a nonzero complex-valued bounded potential* $q = q(x)$ *and a non-trivial complex valued solution* $u = u(x)$ *of*

$$\Delta u = q(x)u, \qquad \textit{in } \mathbb{R}^2, \tag{14}$$

with the property that

$$| u(x) | \leq C \exp(- | x |^{4/3}), \qquad \forall\, x \in \mathbb{R}^2 \tag{15}$$

for some positive constant $C > 0$.

Remark 1 • *The growth rate* $\exp(-\mid x \mid^{4/3})$ *is optimal. Indeed, as proved by Meshkov using a Carleman inequality, if the solution decays faster it has to be zero. This is true for all* n *(space dimension) and* N *(size of the elliptic system):*

$$\forall v \in C_0^\infty(\{r > 1\}),$$
$$\tau^3 \int |v|^2 \exp(2\tau r^{4/3}) r^{2-n} dx \leq C \int |\Delta v|^2 \exp(2\tau r^{4/3}) r^{2-n} dx. \quad (16)$$

- *Constructing solutions decaying as* $\exp(-\mid x \mid^{4/3})$ *for scalar equations is an interesting open problem. The construction by Meshkov is based on a decomposition of* $\mathbb{R}^n$ *into concentric and divergent annulae in which the frequency of oscillation of harmonics* *increases and, simultaneously, the modulus of the solution diminishes. For doing that the particular structure of the spherical harmonics* $r^{-k}\exp(-ik\theta)$ *and, in particular, the fact that* $|\exp(-ik\theta)| = 1$ *plays a key role.*

- *In* $1-d$ *an ODE argument shows that the decay rate is at most exponential. Thus, the superexponential decay for the elliptic problem can not be obtained and the optimality of the parabolic observability inequality can not be proved in this way.*

Null control $\Longrightarrow$ Approximate control

The property of null controllability, even though, apparently, it only guarantees that we can reach the state $\{0\}$, in fact it implies that a dense set of data is reachable. This can be viewed in two steps:

- Step 1: Using the linearity of the system it can be shown that all $u^1 \in S(T)(L^2(\Omega))$, the range of the uncontrolled semigroup, is reachable.

- Step 2: The set $S(T)(L^2(\Omega))$ is dense in $L^2(\Omega)$. This property, by duality, is equivalent to the property of backward uniqueness.

CONSEQUENCES ON THE CONTROL OF NONLINEAR SYSTEMS

Consider semilinear parabolic equation of the form

$$\begin{cases} y_t - \Delta y + g(y) = f 1_\omega & \text{in} \quad \Omega \times (0,T) \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{17}$$

Theorem 3 *(E. Fernández-Cara + EZ, Annales IHP, 2000) The semilinear system is null controllable if*

$$g(s)/ \mid s \mid \log^{3/2} \mid s \mid \to 0 \text{ as } \mid s \mid \to \infty. \tag{18}$$

Note that blow-up phenomena occur if

$$g(s) \sim | s | \log^p(1+ | s |), \text{ as } | s | \to \infty$$

with $p > 1$. Thus, in particular, weakly blowing-up equations may be controlled.

On the other hand, it is also well known that blow-up may not be avoided when $p > 2$ and then control fails.

Note that in the control process the propagation of energy in the x direction plays a key role. When viewing the underlying elliptic problem $\Delta y + g(y)$ a a second order differential equation in x we see how the critical exponent $p = 2$ arises. For $p > 2$ concentration in space may occur so that the control may not avoid the blow-up to occur outside the control region ω.

OBSERVABILITY and GEOMETRY

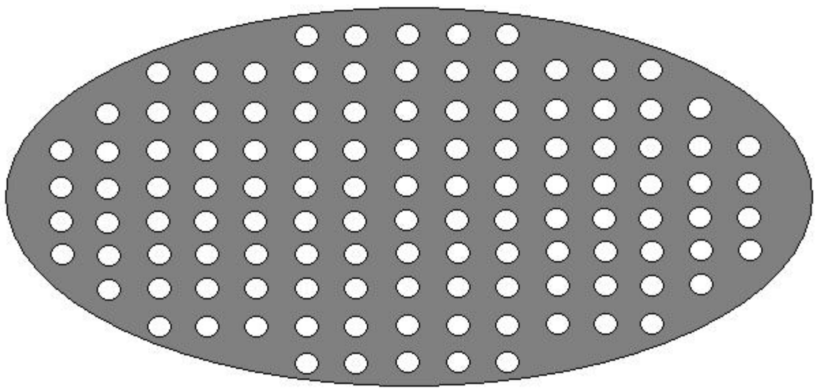
In the absence of potential, the Carleman inequality yields the following observability estimate for the solutions of the heat equation:

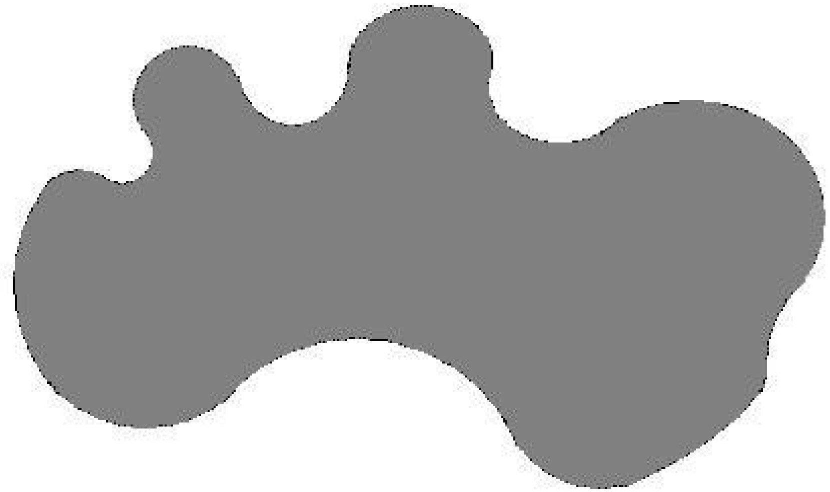
$$\int_0^\infty \int_\Omega e^{\frac{-A}{t}} \varphi^2 dxdt \leq C \int_0^\infty \int_\omega \varphi^2 dxdt.$$

Open problem: Characterize the best constant A in this inequality:

$$A = A(\Omega, \omega).$$

– The Carleman inequality approach allows establishing some upper bounds on A depending on the properties of the weight function. But this does not give a clear path towards the obtention of a sharp constant.





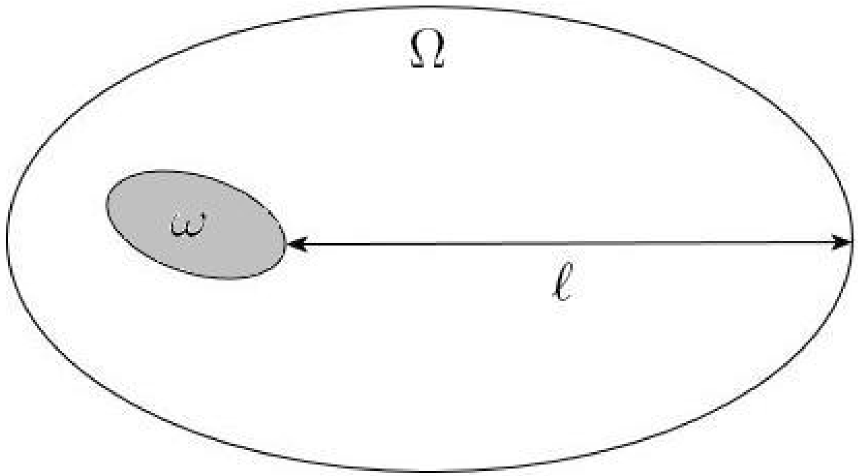
– By inspection of the heat kernel one can see that for the inequality to be true one needs (L. Miller, 2003)

$$A > \exp \ell^2/4$$

where ℓ is the length of the largest segment in $\Omega - \omega$.

Recall that:

$$G(x,t) = (4\pi t)^{-n/2} \exp\left(\frac{-|x|^2}{4t}\right).$$



Characterizing the best constant A is an interesting open problem. It is even open in $1-d$. The best known constant is due to G. Tenenbaum & M. Tucsnak.

G. Tenenbaum & M. Tucsnak. New blow-up rates for fast controls of Schrödinger and heat equations, 2006.

The spectral approach

Lebeau and Robbiano[**] proposed (1996) a spectral proof of the null controllability that, by duality, yields observability inequalities too. The key ingredient is the following estimate on the linear independence of restrictions of eigenfunctions of the laplacian:

Theorem 4 (Lebeau + Robbiano, 1996)

Let Ω be a bounded domain of class C^∞. For any non-empty open subset ω of Ω there exist $B, C > 0$ such that

$$Ce^{-B\sqrt{\mu}} \sum_{\lambda_j \leq \mu} | a_j |^2 \leq \int_\omega \left| \sum_{\lambda_j \leq \mu} a_j \psi_j(x) \right|^2 dx \quad (19)$$

[**] G. Lebeau and L. Robbiano, "Contrôle exact de l'équation de la chaleur", *Comm. P.D.E.*, **20** (1995), 335-356.

for all $\{a_j\} \in \ell^2$ *and for all* $\mu > 0$.

Geometric open problem: To characterize the best constant $B = B(\Omega, \omega)$.

Is the constant B in this spectral inequality related to the best constant $A > 0$ in the parabolic one?

By inspection of the gaussian heat kernel it can be shown that this estimate, i. e. the degeneracy of the constant as $\exp(-B\sqrt{\mu})$ for some $B > 0$, is sharp even in $1-d$.

Although the constant $Ce^{-B\sqrt{\mu}}$ degenerates exponentially as $\mu \to \infty$, it is important that it does it exponentially on $\sqrt{\mu}$. The strong dissipativity ($e^{-\mu t}$) of the heat equation allows compensating this degeneracy and to control the system, after all.

* As a consequence of the spectral estimate one can prove that the observability inequality holds for solutions with initial data in $E_\mu = \text{span}\left\{\psi_j\right\}_{\lambda_j \le \mu}$, the constant being of the order of $\exp\left(B\sqrt{\mu}\right)$. This shows that the projection of solutions over E_μ can be controlled to zero with a control of size $\exp\left(B\sqrt{\mu}\right)$. Thus, when controlling the frequencies $\lambda_j \le \mu$ one increases the $L^2(\Omega)$-norm of the high frequencies $\lambda_j > \mu$ by a multiplicative factor of the order of $\exp\left(B\sqrt{\mu}\right)$.

  This holds in fact for all evolution PDE allowing a Fourier decomposition on the basis of the eigenfunctions of the laplacian.

* However, solutions of the heat equation without control ($f = 0$) and such that the projection of the initial data over E_μ

vanishes, decay in $L^2(\Omega)$ at a rate of the order of $\exp(-\mu t)$. This can be easily seen by means of the Fourier series decomposition of the solution.

Thus, if we divide the time interval $[0, T]$ in two parts $[0, T/2]$ and $[T/2, T]$, we control to zero the frequencies $\lambda_j \leq \mu$ in the interval $[0, T/2]$ and then allow the equation to evolve without control in the interval $[T/2, T]$, it follows that, at time $t = T$, the projection of the solution u over E_μ vanishes and the norm of the high frequencies does not exceed the norm of the initial data u^0:

$$\exp(B\sqrt{\mu}) \exp(-T\mu/2) << 1.$$

This argument allows to control to zero the projection over E_μ for any $\mu > 0$ but not the whole solution.

* To control the whole solution an iterative argument is needed in which the interval $[0, T]$ has to be decomposed in a suitably chosen sequence of subintervals $[T_k, T_{k+1})$ and the argument above is applied in each subinterval to control an increasing range of frequencies $\lambda \leq \mu_k$ with $\mu_k \to \infty$ at a suitable rate.

When the evolution equation under consideration allows a Fourier series decomposition this argument is extremely useful. For instance it allows controlling equations of the form

$$y_t + (-\Delta y)^\alpha = 0,$$

with $\alpha > 1/2$. This is sharp since null control property fails for $\alpha = 1/2$ (S. Micu & E. Z, 2003; L. Miller, 2005). [††] This shows that this iterative construction provides sharp results.

[††]S. Micu & E. Z. On the controllability of a fractional order parabolic equation

The method also applies for time-discrete numerical approximations (C. Zheng, 2006[‡‡]):

$$\frac{y^{k+1} - y^k}{\Delta t} - \Delta y^{k+1} = 0.$$

SIAM J. Cont. Optim., 44(6) (2006) 1950-1972; L. Miller, On the controllability of anomalous diffusions generated by the fractional Laplacian, Math. Control Signals Systems, 2006.

[‡‡]C. Zheng, Controllability of the time-discrete heat equation, 2006, to appear.

OTHER IMPORTANT ISSUES: OPEN PROBLEMS

* Heat equation with non-smooth coefficients on the principal part. Possibly piecewise constant coefficients.

  Recently the observability has been proved in $1-d$ for bounded measurable coefficients, without any other regularity assumption by G. Alessandrini & L. Escauriaza, 2006.

  In higher space dimensions Carleman inequalities can be applied with $W^{1,\infty}$ coefficients.

G. Alessandrini & L. Escauriaza, Null-controllability of one-dimensional parabolic equations, ESAIM:COCV, to appear.

* To exploit the possibility that the potential $a = a(x,t)$ depends both on x and t and not only on x to improve the optimality result. Note for instance that Meshkov also constructs in $3-d$ a potential $a(x,t)$ for the heat equation for which solutions decay as $t \to \infty$ with velocity $\exp(-ct^2)$.

* Heat equations on graphs and networks.

  R. DAGER & E. Z. Wave propagation and control in $1-d$ vibrating multi-structures. Springer Verlag. "Mathématiques et Applications", Paris. 2005

3.2. Hyperbolic control implies parabolic one

As we have seen, the geometric requirements needed for the control of the wave equation are not needed in the context of heat equations: Control can be achieved from arbitrarily small subdomains and in an arbitrarily small control time.

It is known that, within the class of equations with time-independent coefficients

Control of the wave equation $\Longrightarrow$ Control of the heat equation in arbitrarily short time

Note however, that this "hyperbolic$\rightarrow$ parabolic principle" does not yield sharp results for the heat equation since, roughly, it only works under the GCC principle.

There are two ways of doing that:

* Spectral methods (D. L. Russell, 1973.)

  The wave equation:

$$\sum_{j\geq 1}(a_j^+ e^{i\sqrt{\lambda_j}t} + a_j^- e^{-i\sqrt{\lambda_j}t})\phi_j(x).$$

  The heat equation:

$$\sum_{j\geq 1} b_j e^{-\lambda_j t}\phi_j(x).$$

D. L. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations. *Studies in Appl. Math.*, **52** (1973), 189–221.

* Kannai transform (Y. Kannai, 1977; K. D. Phung, 2001; L. Miller, 2004)

$$e^{t\Delta}\varphi = \frac{1}{4\pi t}\int_{-\infty}^{+\infty} e^{-s^2/4t} W(s)ds$$

where $W(x, s)$ solves the wave equation with data $(\varphi, 0)$.

This kind of transformation can be used also, for instance, to transfer results on the wave equation into the Schrödinger one. It also applies in an abstract setting to transfer the abstract wave equation $w_{tt} + Aw = 0$ into the heat one $y_t + Ay = 0$. But, for it to apply, the transfer kernel has to have suitable integrability properties. Consequently this transformation can not be used, for instance, all the way around, to transfer the heat equation into the wave one.

Y. Kannai, Off diagonal short time asymptotics for fundamental solutions of diffusion equations, Commun. Partial Differ. Equations 2 (1977), **8**, 781830; K. D. Phung, Observability and control of Schrödinger equations. SIAM J. Control Optim. **40** (1) (2001), 211–230; L. Miller, Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time, *J. Differential Equations,* **204** (1) (2004), 202–226.

Note that none of these methods applies in the context of heat equations with potentials depending both on x and t:

$$u_t - \Delta u + a(x,t)u = 0.$$

Observe also that this method does not yield any counterexample to the control of heat like equations. For instance the high frequency pathologies for wave equations with rough coefficients are damped out by the heat dissipation.

Indeed, the concentration effects for the eigenfunctions ϕ of wave equations with rough coefficients are of the order of $e^{c\sqrt{\lambda}}$. But, for the heat equation solutions are of the form $e^{-\lambda t}\phi(x)$ which compensates this concentration effect.

3.3 Numerical approximation in 1-d

In $1-d$, using Fourier series arguments, one can prove that the null controls for the $1-d$ semi-discrete finite-difference heat equation converge to the null control of the continuous heat equation.

The key ingredient is proving a uniform (with respect to $h > 0$) observability inequality. It is an immediate consequence of the explicit form of the spectrum together with a technical result on series of real exponentials.

A. López & E. Z.. Some new results related to the null controllability of the $1.d$ heat equation. Seminaire X- EDP 1997-1998, Ecole Polytechnique, 1998, VIII 1-22.

Consider the system:

$$\begin{cases} \varphi_j' - \frac{1}{h^2}\left[\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j\right] = 0, & 0 < t < T,\, j = 1, \dots, N \\ \varphi_j(t) = 0, & j = 0,\, N+1,\, 0 < t < T \\ \varphi_j(0) = \varphi_j^0, & j = 1, \dots, N. \end{cases}$$

We are going to show that the following observability inequality holds uniformly on $h > 0$:

$$||\varphi_h(T)||^2 \leq C(T) \int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt.$$

This is due to the very properties of the eigenvalues entering in its Fourier expansion:

$$\lambda_k^h = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right)$$

Consider the class $\mathcal{L}(\xi, M)$ increasing sequences of positive real numbers $\left\{\nu_j\right\}_{j\geq 1}$ such that

$$\begin{aligned} \nu_{j+1} - \nu_j \geq \xi > 0, \quad \forall j \geq 1, \\ \forall \delta > 0, \sum_{k \geq M(\delta)} \frac{1}{\nu_k} \leq \delta. \end{aligned} \tag{20}$$

Here ξ is any positive number and $M : (0, \infty) \to \mathbb{N}$ is a function such that $M(\delta) \to \infty$ as $\delta \to 0$.

Obviously, different values of ξ and M determine different classes of sequences $\mathcal{L}(\xi, M)$.

The following holds:

Proposition 2 *Given a class of sequences $\mathcal{L}(\xi, M)$ and $T > 0$, there exists a positive constant $C > 0$ such that*

$$\int_0^T \left| \sum_{k=1}^{\infty} a_k e^{-\nu_k t} \right|^2 dt \geq \frac{C}{\left[\sum_{k \geq 1} 1/\nu_k \right]} \sum_{k \geq 1} \frac{|a_k|^2 e^{-2\nu_k T}}{\nu_k}, \qquad (21)$$

for all $\{\nu_k\}_{k \geq 1} \in \mathcal{L}(\xi, M)$ and all bounded sequence $\{a_k\}_{k \geq 1}$.

The sequences of eigenvalues

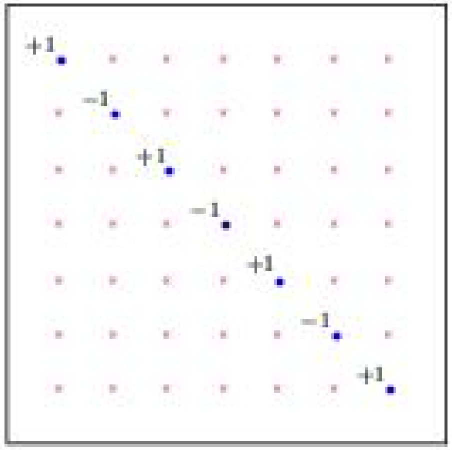
$$\lambda_k^h = \frac{4}{h^2} \sin^2 \left(\frac{k\pi h}{2} \right)$$

of the semi-discrete heat equations belong to the same class $\mathcal{L}(\xi, M)$ for all $h > 0$. Thus, the observability constant C is uniform, i. e. independent of h, for all $T > 0$.

Once this holds, the controls are bounded and converge to the controls of the heat equation.

3.4 Multi-dimensional pathologies

As we have mentioned in the context of the wave and the Schrödinger equation, the classical unique continuation result for the eigenfunctions of the Laplacian guaranteeing that the only one vanishing in an open non-empty subset is the trivial one, fails in the discrete-context.



In view of this one immediately deduces that the unique continuation property fails as well for the corresponding evolution problems, including the heat equation. This shows that for the $2-d$ semi-discrete heat equation when the control subdomain (or subset of the boundary) does not intersect the diagonal the approximate controllability property fails. Null controllability fails as well.

Consequently, for semi-discrete heat equations, in order to get null controllability results the control subset needs to be sufficiently large. In the following section we prove that, when the control acts everywhere on the boundary, the null controllability property holds uniformly on the mesh-size parameter $h > 0$.

3.5 A first result in the multi-dimensional case

We analyze the null controllability of the finite-difference semi-discretization of the heat equation in a bounded domain Ω, by means of boundary controls supported in a subset Γ_0 of the boundary $\partial\Omega$. In the continuous setting null controllability is known to hold in any time $T > 0$.

We first consider the case where Ω is the square domain $\Omega = (0, \pi) \times (0, \pi)$, and the control subdomain is one side of the boundary $\Gamma_0 = \{x_2 = 0.\}$.

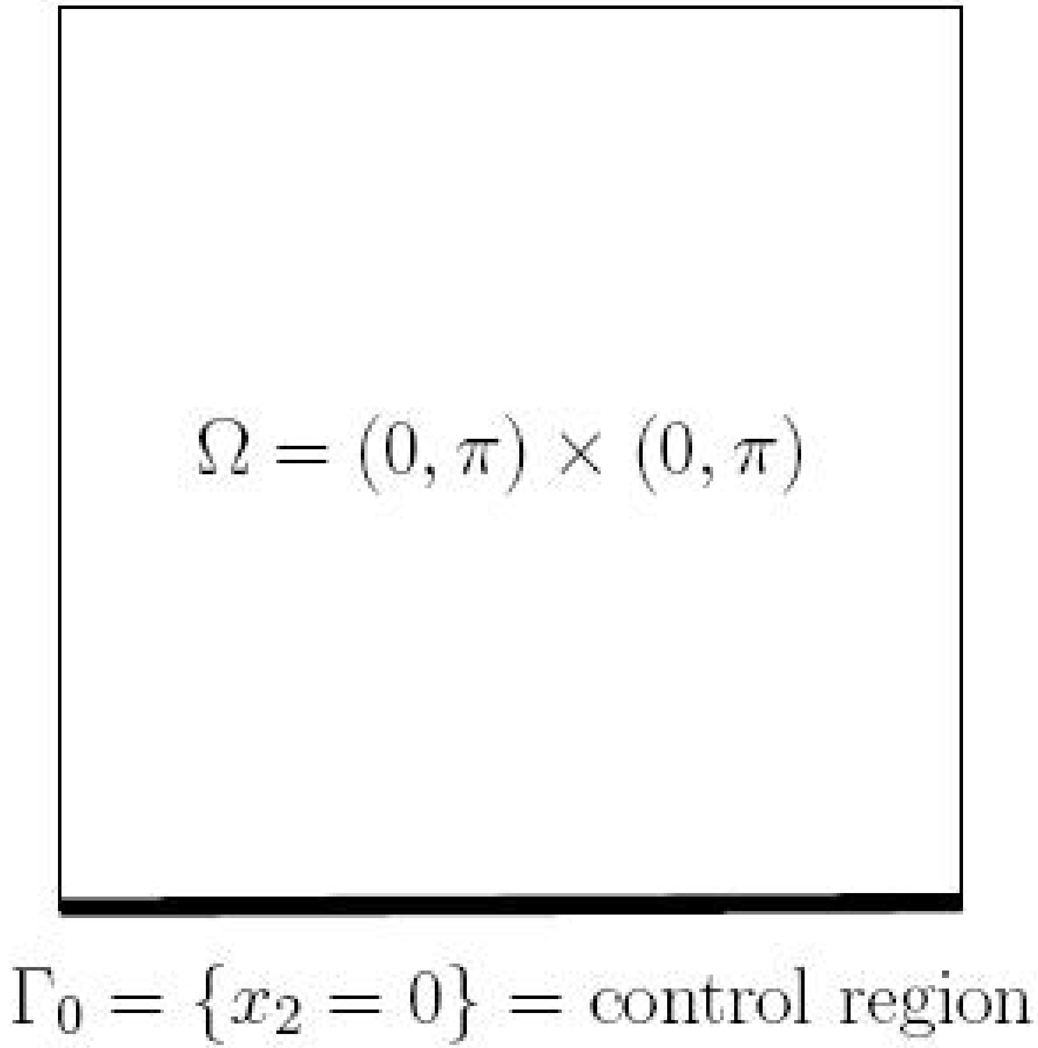
Given $N \in \mathbb{N}$ we set $h = \pi/(N+1)$ and we consider the mesh

$$x_{i,j} = (ih, jh), \quad i, j = 0, ..., N+1, \tag{22}$$

and introduce the finite-difference semi-discretizations:

$$\begin{cases} y'_{j,k} + \frac{1}{h^2}(4y_{j,k} - y_{j+1,k} & -y_{j-1,k} - y_{j,k+1} - y_{j,k-1}) = 0, \\ & (j,k) \in \Omega_h,\ 0 < t < T, \\ y_{j,k} = 0, & (j,k) \in [\partial\Omega \setminus \Gamma_0]_h,\ 0 < t < T, \\ y_{j,0} = v_j, & j = 0, ..., N+1,\ 0 < t < T \\ y_{j,k}(0) = y^0_{j,k}, & (j,k) \in \Omega_h, \end{cases} \tag{23}$$

and


$\Omega = (0,\pi) \times (0,\pi)$
$\Gamma_0 = \{x_2 = 0\}$ = control region

$$
\begin{cases}
\varphi'_{j,k} - \frac{1}{h^2}(4\varphi_{j,k} - \varphi_{j+1,k} & -\varphi_{j-1,k} - \varphi_{j,k+1} - \varphi_{j,k-1}) = 0, \\
 & (j,k) \in \Omega_h,\ 0 < t < T, \\
\varphi_{j,k} = 0, & (j,k) \in [\partial\Omega]_h,\ 0 < t < T \\
\varphi_{j,k}(T) = \varphi^0_{j,k}, & (j,k) \in \Omega_h.
\end{cases}
\tag{24}
$$

Ω_h (resp. $\partial\Omega_h$) is the set of interior (resp. boundary) nodes, and $[\partial\Omega\backslash\Gamma_0]_h$ the set of indices (j,k) so that the corresponding nodes belong to $\partial\Omega \setminus \Gamma_0$. Here and in the sequel $y_{j,k} = y_{j,k}(t)$ (resp. $\varphi_{j,k} = \varphi_{j,k}(t)$) stands for an approximation of the controlled solution y (resp. the adjoint state φ of) at the mesh-points $x_{i,j}$. On the other hand, v_j denotes the control that acts on the semi-discrete system (24) through the subset $[\Gamma_0]_h$ of the boundary. The control does not depend of the index k

since the subset of the boundary $[\Gamma_0]_h$ where the control is being applied corresponds to $k = 0$.

We introduce the vector unknowns and control

$$Y_h = \left(y_{j,k}\right)_{0 \le j,k \le N+1}, \; \Phi_h = \left(\phi_{j,k}\right)_{0 \le j,k \le N+1}, \; V_h = \left(v_j\right)_{1 \le j \le N}, \tag{25}$$

that we shall often denote simply by Y, Φ and V.

The previous systems read:

$$\begin{cases} Y_h' + A_h Y_h = B_h V_h, \\ Y_h(0) = Y_h^0, \end{cases} \tag{26}$$

$$\begin{cases} \Phi_h' - A_h \Phi_h = 0, \\ \Phi_h(T) = \Phi_h^0. \end{cases} \tag{27}$$

We denote by A_h the usual positive-definite symmetric Toeplitz matrix associated with the five-point finite-different scheme we have employed in the discretization of the Laplacian so that

$$(A_hW)_{j,k} = \frac{1}{h^2}(4w_{j,k} - w_{j+1,k} - w_{j-1,k} - w_{j,k+1} - w_{j,k-1}), \quad (28)$$

for the inner nodes. The linear operator B_h in (26) is such that the action of the control v_j enters on those nodes which are neighbors to those of $[\Gamma_0]_h$, i. e. for $k = 1$, so that $[B_hV]_{j,k} = 0$ whenever $2 \leq k \leq N$ but $[B_hV]_{j,1} = -v_j/h^2$.

The null-controllability problem for system (26) reads as follows: Given $Y^0 \in \mathbb{R}^{N+2\times N+2}$ to find $V \in L^2\left(0, T; \mathbb{R}^N\right)$ such that the solution Y of (26) satisfies

$$Y(T) = 0. \quad (29)$$

On the other hand, the problem of observability for system (27) consists in proving the existence of $C > 0$ such that

$$\left|\Phi(0)\right|_h^2 \leq Ch \int_0^T \sum_{j=1}^{N} \left|\frac{\phi_{j,1}}{h}\right|^2 dt \qquad (30)$$

for every solution Φ of (27).

In (30) $\left|\cdot\right|_h$ stands for the scaled Euclidean norm

$$\left|\Phi\right|_h = \left[h^2 \sum_{j,k=0}^{N+1} |\phi_{j,k}|^2\right]^{1/2} \qquad (31)$$

and the right hand side term of inequality (30) represents the discrete version of the L^2-norm of the normal derivative.

The following result holds:

A similar problem can be formulated in general bounded smooth domains Ω. In that case, obviously, the domain Ω needs to be approximated by domains Ω_h whose boundaries are constituted by mesh-points.

In [S. Labbé, S., E. Trélat. Uniform controllability of semidiscrete approximations of parabolic control systems. *Systems & Control Letters*, 55 (2006), no. 7, 597–609] the problem of approximate controllability has been analyzed. There it has been proved that, as the mesh-size tends to zero, the numerical scheme reproduces the controllability properties of the continuous heat equation getting explicit convergence rates and bounds on the number of iterations needed when applied descent algorithms.

Theorem 5 *Let* $T > 0$ *be any positive control time. Let* $y^0 \in L^2(\Omega)$ *and* Y_h^0 *a discrete approximation. Then, the null controls* V_h *for the semi-discrete problem (26) are uniformly bounded, with respect to* h *and converge in* $L^2(\Gamma_0 \times (0, T))$ *towards the null control of the heat equation. The semi-discrete controlled states* Y_h *also converge to the controlled state* y *of the heat equation in* $L^2(0, T; H^{-1}(\Omega))$ *satisfying the null final condition.*

Remark 2 *The result is sharp in what concerns the support* Γ_0 *of the control. It fails when* $[\Gamma_0]_h$ *is replaced by the set of indices* $[\Gamma_0^*]_h$ *in which the first node corresponding to the index* $j = 1$ *is removed.*

The key point of the proof is proving that the observability inequality (30) is uniform with respect to the mesh-size $h > 0$.

The method of proof of the uniform estimate (30) depends heavily on the Fourier decomposition of solutions.

The eigenvalue problem associated with the semi-discrete system (27) is as follows:

$$\begin{cases} \frac{1}{h^2}\left[4w_{j,k} - w_{j+1,k} - w_{j-1,k} - w_{j,k+1} - w_{j,k-1}\right] = \lambda w_{j,k}, & \Omega_h \\ w_{j,k} = 0, & [\partial\Omega]_h. \end{cases} \tag{32}$$

Its spectrum may be computed explicitly:

$$\lambda^{\ell,m}(h) = \frac{4}{h^2}\left[\sin^2\left(\frac{\ell h}{2}\right) + \sin^2\left(\frac{mh}{2}\right)\right] \tag{33}$$

$$W^{\ell,m}(h) = w^{\ell,m}(x)\Big|_{x=(jh,kh),j,k=0,\cdots,N+1} \tag{34}$$

for $\ell, m = 1, ..., N$, where $w^{\ell,m}(x)$ are the eigenfunctions of the continuous Laplacian:

$$w^{\ell,m}(x) = \frac{2}{\pi} \sin(\ell x_1) \sin(m x_2).$$

We have:

* The eigenvectors are the restrictions to the mesh of the eigenfunctions of the Laplacian;

* Eigenvalues converge:

$$\lambda^{\ell,m}(h) \to \lambda^{\ell,m} = \ell^2 + m^2 \text{ as } h \to 0. \tag{35}$$

The eigenvectors $\left\{W^{\ell,m}\right\}_{\ell,m=1,\dots,N}$ constitute an orthonormal basis of $\mathbb{R}^{N\times N}$ with respect to the scalar product

$$\langle f, \widetilde{f}\rangle_h = \left[h^2 \sum_{j,k=1}^{N} f_{j,k}\widetilde{f}_{j,k}\right]^{1/2}, \tag{36}$$

associated with the norm (31).

The solution of the semi-discrete adjoint system (27) can also be easily developed in this basis:

$$\Phi_h(t) = \sum_{\ell,m=1}^{N} a^{\ell,m} e^{-\lambda^{\ell,m}(h)(T-t)} W^{\ell,m} \tag{37}$$

where $\left\{a^{\ell,m}\right\}$ are the Fourier coefficients of the datum at time $t = T$:

$$\Phi_h^0 = \sum_{\ell,m=1}^{N} a^{\ell,m} W^{\ell,m}, \qquad a^{\ell,m} = \langle \Phi_h^0, W^{\ell,m}\rangle_h. \tag{38}$$

Solutions may also be rewritten in the form

$$\Phi_h(t) = \sum_{\ell=1}^{N} \psi^m(t) \otimes \sigma^m, \tag{39}$$

where

$$\sigma^m = \left(\frac{\sqrt{2}}{\sqrt{\pi}} \sin(mkh) \right)_{k=0,\ldots,N+1},$$

so that $W^{\ell,m} = \sigma^\ell \otimes \sigma^m$, and each vector-valued function $\psi^m(t) = \left(\psi_j^m(t)\right)_{j=0,\ldots,N+1}$ is a solution of the $1-d$ semi-discrete problem:

$$\begin{cases} \psi_j' - \frac{[2\psi_j - \psi_{j+1} - \psi_{j-1}]}{h^2} + \mu^m \psi_j = 0, \; j = 1, \ldots, N, \; 0 < t < T \\ \psi_0 = \psi_{N+1} = 0, \qquad 0 < t < T \\ \psi_j(T) = \psi_j^0, \qquad j = 1, \ldots, N, \end{cases} \tag{40}$$

where $\mu^m = \frac{4}{h^2} \sin^2\left(\frac{mh}{2}\right)$.

The observability inequality (30) is equivalent to proving the $1-d$ analogue for (40), uniformly on $m \geq 1$, i.e.

$$\left|\psi(0)\right|_h^2 \leq C \int_0^T \left|\frac{\psi_1}{h}\right|^2 dt, \tag{41}$$

for all ψ^0, ψ being the solution of (40), with a constant $C > 0$ which is independent of m.

This is trivially true since an explicit change of variables reduces these problems to the particular case $\mu^m = 0$.

General domains

The methods of proof of the previous section, based on Fourier series expansions, do not apply to general domains.

However, using a classical argument, based on extending the control domain and then getting the controls as restrictions to the original boundary of the controlled states, one can derive similar results for general domains but provided the controls are supported everywhere on the boundary of the domain.

The problem of determining sharp conditions on the subsets of the boundary so that the semi-discrete systems are uniformly controllable is completely open.

3.6 The Carleman inequalities aproach

The results in the previous section show that null-controllability and observability hold when the control acts everywhere on the boundary of the domain. However, to deal with more general problems (equations with variable coefficients, nonlinear problems,...) it would be convenient to be able to derive these results by more systematic methods. In particular it would be desirable to develop the methodlogy based on Carleman inequalities.

Anyhow, the counterexample of the eigenvector for the discrete Laplacian concentrated in the square shows that, for the observability inequality to be true, the high frequency components have to be filtered out.

The following result could be true: Given any subdomain ω of the square ω, the observability inequality holds for all eigenvectors of the discrete Laplacian, uniformly on the mesh-size parameter $h > 0$, within the class of eigenvectors corresponding to eigenvalues

$$\lambda \leq \frac{c(\Omega, \omega)}{h^2},$$

for a suitable geometric constant.

In fact this result could well be true for all domains Ω and ω.

This is an interesting open problem.