# Control and asymptotics for some free boundaries[1]

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[1] Based on joint work with Borjan Geshovski

# Free boundary problems

# Free boundary problems

- **Unknowns** are the state and a part of the boundary
- The (transient) prototype: *one-phase Stefan problem*

$$\begin{cases} T_t - T_{xx} = f & \text{for } t \geq 0,\ 0 < x < s(t) \\ s'(t) = -T_x(s(t), t) & \text{for } t \geq 0 \\ T(0,t) = T(s(t),t) = 0 & \text{for } t \geq 0 \\ T(x,0) = T_0(x) & \text{for } 0 < x < s_0, \end{cases}$$

  where $(T, s)$ are unknown, while $s_0 \geq 0$ and $(f, T_0)$ are given
- Model for the melting of a block of ice inside a container filled with water
- The *Stefan condition* describes the motion of the melting interface.

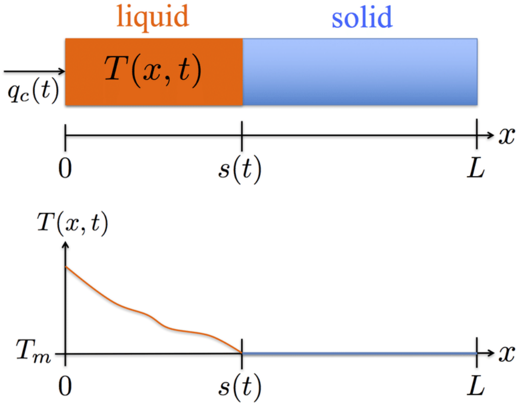


Figure: $T$ is the temperature and $s$ is the melting front. In our case, $q_c, T_m \equiv 0$.

# Control of parabolic problems

# Basics on parabolic equations

- The canonical example is the *heat equation*

$$\begin{cases} y_t - \Delta y = f\,\mathbb{1}_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y = y_0 & \text{in } \Omega, \end{cases} \tag{1}$$

  where $\Omega \subset \mathbb{R}^d$ is a bounded domain with $C^2$ boundary, $\omega \subset \Omega$, $\mathbb{1}$ is the indicator function, and $(f, y_0)$ are given
- *Smoothing effect*: $f\,\mathbb{1}_\omega \equiv 0$ on $\Omega \setminus \omega \Longrightarrow y(t,\cdot) \in C^\infty(\Omega \setminus \omega)$ for $t > 0$, even if $y_0 \in L^2(\Omega)$.

# Controllability of the heat equation

There are multiple concepts of controllability, the "basic" one being

## Definition (Exact controllability at time $T > 0$)

For any $y_0, y_1 \in L^2(\Omega)$, there exists $f \in L^2((0, T) \times \Omega)$ such that the solution $y$ to (1) satisfies

$$\boxed{y(T, x) = y_1(x) \quad \text{for } x \in \Omega.}$$

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- Smoothing effect $\Rightarrow$ if $\omega \neq \Omega$, then (1) is not exactly controllable.
- Can we steer $y$ to specific targets, such as $y_1 \equiv 0$? This is the problem of *null-controllability*.

# Null-controllability of the heat equation

*Hilbert Uniqueness Method*: null-controllability at time $T > 0$ is equivalent to: $\exists C = C(T) > 0$ such that for all $\varphi^T \in L^2(\Omega)$,

$$\boxed{\int_\Omega |\varphi(x,0)|^2 dx \leq C \int_0^T \int_\omega |\varphi(x,t)|^2 dxdt,}$$

where $\varphi$ is the solution to the *adjoint problem*

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } (0,T) \times \Omega \\ \varphi = 0 & \text{on } (0,T) \times \partial\Omega \\ \varphi(T,\cdot) = \varphi^T & \text{in } \Omega. \end{cases}$$

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This is called an *observability inequality*, and we say that the adjoint problem is (final-state) *observable*.

# Null-controllability of the heat equation

How to prove the observability inequality?

- Fourier techniques: if eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ known, roughly check if $\lambda_{k+1}-\lambda_k>0$ for all $k\in\mathbb{N}$ (biorthogonals of Fattorini & Russell 70s)
- Carleman inequalities (Fursikov-Imanuvilov '96, Lebeau-Robbiano '95, Zuazua-Fdez Cara '00)

Remark: Distributed $\Longrightarrow$ boundary null-controllability (observability) for parabolic problems.
Other ways for proving controllability include flatness techniques[2] and characterization of the reachable space of the heat equation[3], or tranmutation techniques[4].

[2] Rouchon, Rosier et al. 2010s
[3] Tucsnak et al., JEMS '18
[4] Miller JFA '05, Ervedoza-E.Z. ARMA '11

# Null-control of the Stefan problem

E.Férnandez-Cara et al. (2016): null-controllability roughly by means of the scheme:

1. Fix $s \in C^1([0, T])$, and consider

$$\begin{cases} y_t - y_{xx} = f\mathbb{1}_\omega & \text{for } t \geq 0,\ 0 < x < s(t) \\ y(0,t) = y(s(t),t) = 0 & \text{for } t \geq 0 \\ y(x,0) = y_0(x) & \text{for } 0 < x < s_0. \end{cases} \tag{2}$$

   Notice that we have removed the Stefan condition.

2. Prove that (2) is null-controllable: HUM + Carleman inequality.
3. Transfer this knowledge to the free boundary problem by means of a Schauder fixed-point theorem applied to the map

$$\Lambda : s(t) \mapsto s_0 - \int_0^t y_x(s(\tau),\tau)d\tau.$$

# A different strategy[5]

Liu, Takahashi and Tucsnak, COCV '13: null-controllability for

$$\begin{cases} v_t - v_{xx} + v v_x = 0 & \text{for } t \geq 0,\ x \in (-1,1) \setminus \{h(t)\} \\ v(-1,t) = 0, v(1,t) = u(t) & \text{for } t \geq 0 \\ h'(t) = v(h(t),t) & \text{for } t \geq 0 \\ h''(t) = [v_x](h(t),t) & \text{for } t \geq 0 \\ h(0) = h_0, \quad h'(0) = h_1, & \\ v(x,0) = v_0(x), & \text{for } x \in (-1,1) \setminus \{h_0\}. \end{cases}$$

- Model for the motion of a single particle in a viscous fluid occupying the pipe $(-1,1)$
- $v$ represents the fluid velocity and $h$ the position of the particle
- Null-controllability result includes $h(T) = 0$, $h'(T) = 0$.
- Control acts only on one boundary.

[5] Used in control by Imanuvilov-Takahashi JMPA '08, Tucsnak et al. '13, '14, Geshkovski & E.Z. '19 for control of both PDE and ODE.

1. For $t \geq 0$, change of variable to fix the domain

$$\eta(\cdot, t) : (-1, 1) \setminus \{h(t)\} \to (-1, 1) \setminus \{0\}$$

yielding a nonlinear problem written in Cauchy-form

$$\begin{cases} \dot{z}(t) = Az(t) + B\hat{u}(t) + N \begin{bmatrix} z \\ h \end{bmatrix} \\ h'(t) = Cz(t) \\ z(0) = z_0 \\ h(0) = h_0. \end{cases}$$

2. Consider the linear problem: replace $N \begin{bmatrix} z \\ h \end{bmatrix}$ by $f$
3. Prove null-control. of the linear problem with $f \equiv 0$ using parabolic techniques
4. Transfer null-control. result to problem with $f \not\equiv 0$ if $f$ has decay properties (called *source term method*)
5. $N$ is a contraction $\Longrightarrow$ Banach's fixed-point.

# Asymptotics

# Asymptotics

Fix domain + Hadamard linearization + Banach Fixed point is standard strategy for analysis of free boundary problems in strong form[6]:

$$\begin{cases} u_t - u_{xx} + uu_x = 0 & \text{for } t \geq 0,\ x \in \mathbb{R} \setminus \{h(t)\} \\ h'(t) = u(h(t), t) & \text{for } t \geq 0 \\ h''(t) = [u_x](h(t), t) & \text{for } t \geq 0 \\ h(0) = h_0, \quad h'(0) = h_1, & \\ u(x, 0) = u_0(x), & \text{for } x \in \mathbb{R} \setminus \{h_0\}. \end{cases} \tag{3}$$

What is the asymptotic behaviour of (3)? Scaling arguments.

[6] Vazquez-EZ, Comm PDE '03 M3AS '05, Otto et al. JDE '08, Masmoudi et al. ARMA '15

Theorem (Vazquez-E.Z. '03)

*Let* $u_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ *and* $h_0, h_1 \in \mathbb{R}$*. Then*

$$t^{(1-1/p)/2} \|u(t) - \tilde{u}(t)\|_{L^p(\mathbb{R})} \to 0 \quad \text{as} \;\; t \to \infty \qquad (4)$$

*for all* $1 \le p \le \infty$ *where* $\tilde{u}(x,t) = t^{-1/2} f_M(x/\sqrt{t})$ *is the self-similar solution of Burgers' with mass* $M$ *given by* $M = \int_{\mathbb{R}} u_0(x)dx + h_1$*.*

Question: Can we locate the asymptotic position $h(t)$ and velocity $h'(t)$ of the particle?

## Theorem (Vazquez-E.Z. '03)

*Under the conditions in the above theorem, if $M > 0$ then*

$$t^{-1/2}|h(t) - c\sqrt{t}| \ \to 0 \quad \text{as} \ \ t \to \infty \tag{5}$$

*where $c > 0$ is uniquely determined by the equation $f_M(c) = c/2$.*
*Moreover, we have a precise estimate of the particle speed:*

$$t^{1/2}|h'(t) - \frac{c}{2\sqrt{t}}| \to 0 \quad \text{as} \ \ t \to \infty. \tag{6}$$