Control of Kolmogorov model

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In our article in collaboration with Karine Beauchard

K. Beauchard and E. Z., Some controllability results for the 2D Kolmogorov equation, Ann. I. H. Poincaré ? AN 26 (2009) 1793?1815.

we addressed the null control of the Kolmogorov equation:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial^2 f}{\partial v^2} = u(t,x,v)1_\omega(x,v), (x,v) \in \mathbf{R}_x \times \mathbf{R}_v, t \in (0,+\infty). \quad (1)$$

Consider the particular case where where $\omega = \mathbf{R}_x \times \Big[\mathbf{R}_v - [a,b]\Big]$.

Equivalently, one may address the following observability inequality for the adjoint system:

$$\begin{cases} \frac{\partial g}{\partial t} - v\frac{\partial g}{\partial x} - \frac{\partial^2 g}{\partial v^2} = 0, (x,v) \in \mathbf{R}_x \times \mathbf{R}_v, t \in (0,T), \\ g(0,x,v) = g_0(x,v), (x,v) \in \mathbf{R}_x \times \mathbf{R}_v. \end{cases} \tag{2}$$

$$\int_{\mathbf{R}_x \times \mathbf{R}_v} |g(T,x,v)|^2 dxdv \leqslant C \int_0^T \int_\omega |g(t,x,v)|^2 dxdvdt.$$

Theorem 1 *(K. Beauchard and E. Z.)*

In the particular case where $\omega = \mathbf{R}_x \times \left[\mathbf{R}_v - [a,b]\right]$ *the observability inequality holds for the adjoint system and the Kolmogorov equation is null controllable.*

Ideas of the proof:

- Fourier transform in v:

$$\begin{cases} \frac{\partial \widehat{f}}{\partial t}(t,\xi,v) + i\xi v \widehat{f}(t,\xi,v) - \frac{\partial^2 \widehat{f}}{\partial v^2}(t,\xi,v) = \widehat{u}(t,\xi,v) 1_{\mathbb{R}-[a,b]}(v), \\ \widehat{f}(0,\xi,v) = \widehat{f}_0(\xi,v). \end{cases} \quad (3)$$

- Decay:

$$\left| \widehat{f}(t,\xi,.) \right|_{L^2(\mathbb{R})} \leqslant \left| \widehat{f}_0(\xi,.) \right|_{L^2(\mathbb{R})} e^{-\xi^2 t^3/12}, \forall \xi \in \mathbb{R}, \forall t \in \mathbb{R}_+. \quad (4)$$

- Control depending on the parameter ξ with cost

$$e^{C(T) \max\{1, \sqrt{|\xi|}\}}.$$

The exponentially large cost of control for high frequencies is compensated by the exponential (and stronger) decay rate.

Open problem # 4.1: Similar results hold for other geometries of control sets?

Open problem # 4.2: What about more general classes of hypoelliptic equations?

Open problem # 4.3: May Carleman inequalities be applied directly on the Kolmogorov system without using Fourier transform?

Open problem # 4.4: How are related the notions of hypoellipticity and hypocoercivity with the property of null controllability.

See also:

E. ZUAZUA, "Controllability and Observability of Partial Differential Equations: Some results and open problems", in *Handbook of Differential Equations: Evolutionary Equations, vol. 3*, C. M. Dafermos and E. Feireisl eds., Elsevier Science, 2006, pp. 527-621.