Control of viscous Hamilton-Jacobi equations[1]

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[1] Joint work with A. Porretta, Null controllability of viscous Hamilton-Jacobi equations, Annales IHP, Analyse non linéaire, 29 (2012), pp. 301-333.

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The control problem

Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

$$\begin{cases} u_t - \Delta u = v 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

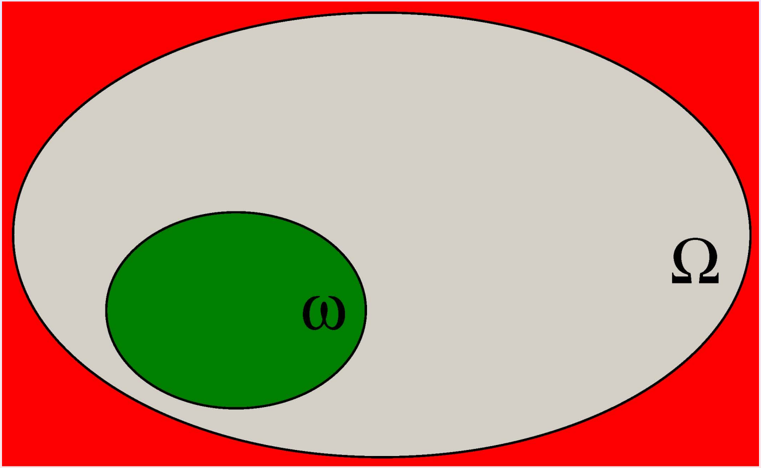
1_ω denotes the characteristic function of the subset ω of Ω where the control is active.

We assume that $u^0 \in L^2(\Omega)$ and $v \in L^2(Q)$ so that (1) admits an unique solution

$$u \in C\left([0, T]; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

$$u = u(x,t) = solution = state, \; v = v(x,t) = control$$

Goal: To produce prescribed deformations on the solution u by means of suitable choices of the control function v.


ω
Ω

The following result is by now well-known.[2]

Theorem

For every bounded domain Ω, any open non-empty subset ω, $T > 0$ and initial datum $u^0 \in L^2(\Omega)$ there exists a control $v \in L^2(\omega \times (0, T))$ such that the solution $u = u(x,t)$ satisfies

$$u(T) \equiv 0.$$

- This result is in agreement with intuition according to which actions in ω propagate instantaneously everywhere in Ω.
- The proof combines duality arguments and Carleman inequalities.

[2] Extensive literature starting with the pioneering work "H. O. Fattorini and D. L. Russell. Exact controllability theorems for linear parabolic equations in one space dimension. Arch. Rational Mech. Anal., 43:272–292, 1971". Later continued by G. Lebeau and L. Robbiano, A. Fursikov and O. Imanuvilov,...

The control can be obtained by minimizing the functional[3]

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) u^0 dx \tag{2}$$

over the class of solutions of the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{3}$$

The coercivity of the functional J requires the following observability inequality:

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{4}$$

[3] J. L. Lions, Remarks on approximate controllability, J. d'Analyse Mathématique, Vol. 59, N. 1 (1992), 103-116.

This estimate was proved by Fursikov and Imanuvilov (1996) using Carleman inequalities. In fact the same proof applies for equations with smooth (C^1) variable coefficients in the principal part and for heat equations with lower order potentials:

$$\begin{cases} \varphi_t - \Delta\varphi + a\varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(0,x) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{5}$$

Theorem

(Fursikov+Imanuvilov, 1996, E. Fernández-Cara+E. Z., 2000) *Assume that* ω *is an open non-empty subset of* Ω*. Then, there exists a constant* $C = C(\Omega,\omega) > 0$*, depending on* Ω *and* ω *but independent of* T*, the potential* $a = a(t,x)$ *and the solution* φ *such that*

$$\| \varphi(T) \|^2_{(L^2(\Omega))^N} \le \exp\left(C\left(1 + \frac{1}{T} + T \| a \|_\infty + \| a \|_\infty^{2/3}\right)\right) \int_0^T \int_\omega |\varphi|^2. \tag{6}$$

- No information on the initial datum is used.
- This is a sidewise energy estimate.
- The problem under consideration is ill-posed.
- We recover information on the solution at the final time but not at $t = 0$ where an exponential boundary layer emerges.

Sketch of the proof

Introduce a function $\eta^0 = \eta^0(x)$ such that:

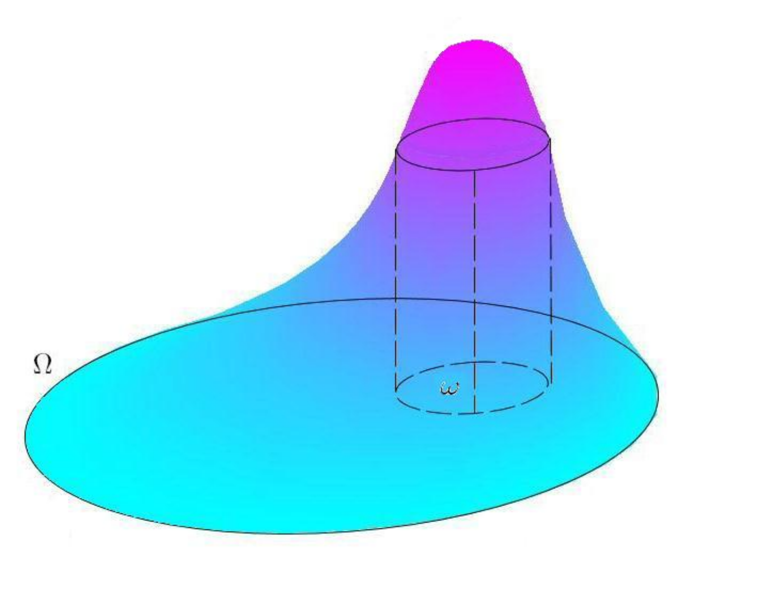
$$\begin{cases} \eta^0 \in C^2(\bar{\Omega}) \\ \eta^0 > 0 \quad \text{in} \quad \Omega,\ \eta^0 = 0 \quad \text{in} \quad \partial\Omega \\ \nabla \eta^0 \neq 0 \quad \text{in} \quad \overline{\Omega \setminus \omega}. \end{cases} \tag{7}$$

In some particular cases, for instance when Ω is star-shaped with respect to a point in ω, it can be built explicitly without difficulty. But the existence of this function is less obvious in general, when the domain has holes or its boundary oscillates, for instance.
Let $k > 0$ such that $k \geq 5 \max_{\bar{\Omega}} \eta^0 - 6 \min_{\bar{\Omega}} \eta^0$ and let

$$\beta^0 = \eta^0 + k, \bar{\beta} = \frac{5}{4} \max \beta^0,\ \rho^1(x) = e^{\lambda\bar{\beta}} - e^{\lambda\beta^0}$$

with $\lambda, \bar{\beta}$ sufficiently large. Let be finally

$$\gamma = \rho^1(x)/(t(T-t)); \rho(x,t) = \exp(\gamma(x,t)).$$


Ω
ω

The following Carleman inequality holds:

Proposition

(Fursikov + Imanuvilov, 1996)
There exist positive constants $C_*, s_1 > 0$ *such that*

$$\begin{aligned}
&\frac{1}{s}\int_Q \rho^{-2s} t(T-t)\left[|q_t|^2 + |\Delta q|^2\right] dxdt \\
&+s\int_Q \rho^{-2s} t^{-1}(T-t)^{-1}|\nabla q|^2 dxdt + s^3\int_Q \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt \\
&\leq C_*\left[\int_Q \rho^{-2s}|\partial_t q - \Delta q|^2 dxdt + s^3\int_0^T\int_\omega \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt\right]
\end{aligned} \tag{8}$$

for all $q \in Z$ *and* $s \geq s_1$.
Moreover, C_* *depends only on* Ω *and* ω *and* s_1 *is of the form*

$$s_1 = s_0(\Omega,\omega)(T + T^2).$$

This yields:

$$\| \varphi(T) \|^2_{(L^2(\Omega))^N} \leq \exp\left(C\left(1 + \frac{1}{T} + T \| a \|_\infty + \| a \|_\infty^{2/3}\right)\right) \int_0^T \int_\omega |\varphi|^2. \tag{10}$$

Three different terms have to be distinguished on the observability constant on the right hand side:

$$C_1^*(T, a) = \exp\left(C\left(1 + \frac{1}{T}\right)\right), \quad C_2^*(T, a) = \exp(CT \| a \|_\infty), \tag{11}$$

$$C_3^*(T, a) = \exp\left(C \| a \|_\infty^{2/3}\right).$$

The role of the first two constants is clear:

• The first one $C_1^*(T, a) = \exp\left(C\left(1 + \frac{1}{T}\right)\right)$ takes into account the increasing cost of making continuous observations as T diminishes.

• The second one $C_2^*(T, a) = \exp(CT \| a \|_\infty)$ is due to the use of Gronwall's inequality to pass from a global estimate in (x, t) into an estimate for $t = T$.

• $2/3 \in [1/2, 1]$!!!!!!!!!!!!!!:

Theorem

(Th. Duyckaerts, X. Zhang and E. Z., Ann. IHP, 2008)
The third constant $C_3^*(T, a)$ is sharp in the range

$$\| a \|_\infty^{-2/3} \lesssim T \lesssim \| a \|_\infty^{-1/3}, \tag{12}$$

for systems $N \geq 2$ and in more than one dimension $n \geq 2$.

The proof is based on the following Theorem by V. Z. Meshkov, 1991.

Theorem

Assume $n = N = 2$. Then, there exists a nonzero complex-valued bounded potential $q = q(x)$ and a non-trivial complex valued solution $u = u(x)$ of

$$\Delta u = q(x) u, \qquad \text{in } \mathbb{R}^2, \tag{13}$$

with the property that

$$| u(x) | \leq C \exp(- | x |^{4/3}), \qquad \forall x \in \mathbb{R}^2. \tag{14}$$

Observability and geometry

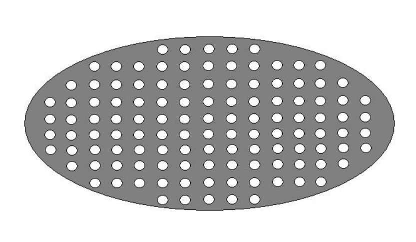
In the absence of potential, the Carleman inequality yields the following observability estimate for the solutions of the heat equation:

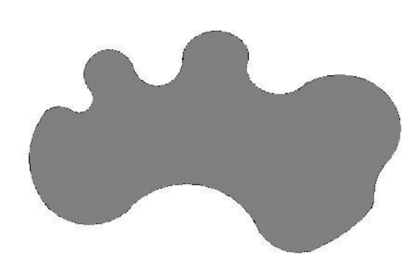
$$\int_0^\infty \int_\Omega e^{\frac{-A}{t}} \varphi^2 dxdt \leq C \int_0^\infty \int_\omega \varphi^2 dxdt.$$

Open problem: Characterize the best constant A in this inequality:

$$A = A(\Omega, \omega).$$

The Carleman inequality approach allows establishing some upper bounds on A depending on the properties of the weight function. But this does not give a clear path towards the obtention of a sharp constant.





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Consider the same problem of null controllability for the semilinear equation

$$u_t - \Delta u + f(x, u, \nabla u) = v\chi_\omega$$

The system is controllable in any time $T > 0$ if the nonlinearity f fulfills a mild growth condition[4]:

$$|f(x, u, \xi)| \leq C\left(|u|(\log(1 + |u|))^\gamma + |\xi|(\log(1 + |\xi|))^\alpha\right)$$

with

$$\gamma < \frac{3}{2}; \quad \alpha < \frac{1}{2}$$

Ex: $f(x, u, \nabla u) = |\nabla u|^q$ with $q \leq 1$ is OK.

Typical method: fixed point argument + controllability of the linearized equation.

[4] See [Doubova-Fernández Cara-González Burgos-Zuazua '02] (general case), [Fernández Cara-Zuazua '00] (case $f = f(x, u)$), previous results in [Fursikov-Imanuvilov '96], [Lebeau-Robbiano '95], [Barbu '00].

- If $f = f(x, u)$ has fast growth and the bad sign, null controllability may fail because of blow-up phenomena.
  Namely, if

$$-f(x, u) \geq c\,|u|(\log(1 + |u|))^{\gamma} \quad \text{with } \gamma > 2$$

  there exist initial data which can never be controlled.
  ([Fernandez Cara-Zuazua '00]

  - Solutions blow-up independently of the control v (while controllability would imply global existence).
  - In the space-like direction the equation reads $-u_{xx} + f(u) = 0$ and nonlinear localization phenomena arise as soon as $f(s) >> |s| \log^2(|s|)$ as $|s| \to \infty$.

Introduce a function $\rho \in \mathcal{D}(\Omega), \rho \not\equiv 0$ such that

$$\rho = 0 \text{ in } \omega; \int_\Omega \rho dx = 1. \tag{15}$$

Then

$$\frac{d}{dt}\left(-\int_\Omega \rho y dx\right) = -\int_\Omega y \Delta \rho dx + \int_\Omega f(\mid y \mid)\rho dx. \tag{16}$$

We assume for the moment that, where f^* is the convex conjugate of f,

$$\rho f^*(2\Delta\rho/\rho) \in L^1(\Omega). \tag{17}$$

Then, by Young's inequality we have

$$\begin{aligned} \left|\int_\Omega y \Delta \rho dx\right| &\leq \int_\Omega \mid y \mid \left|\frac{\Delta \rho}{\rho}\right| \rho dx \\ &\leq \frac{1}{2}\int_\Omega f(\mid y \mid)\rho dx + \frac{1}{2}\int_\Omega f^*(2 \mid \Delta\rho \mid /\rho)\rho dx. \end{aligned} \tag{18}$$

Then, (16) may be rewritten as

$$\frac{d}{dt}\left[-\int_\Omega y \rho dx\right] \geq -k_1 + \frac{1}{2}\int_\Omega f(\mid y \mid)\rho dx.$$

By Jensen's inequality we have

$$f\left[\int_\Omega \mid y \mid \rho dx\right] \leq \int_\Omega f(\mid y \mid)\rho dx.$$

Consequently, if $z(t) = -\int_\Omega y(x,t)\rho(x)dx$, we have

$$z'(t) \geq -k_1 + f(\mid z \mid). \tag{19}$$

This shows blow-up regardless of the value of the control v.
What about the assumption?

$$\rho f^*(2\Delta\rho/\rho) \in L^1(\Omega). \tag{20}$$

It is easy to check that, if $f \sim |s| \log^p(|s|)$, then

$$f^*(s) \sim p \mid s \mid^{1-1/p} \exp\left(\mid s \mid^{1/p}\right), \text{ as } s \to \infty. \tag{21}$$

If $\rho(x) = \exp(-x^{-k})$,

$$\frac{\mid \rho''(x) \mid}{\rho(x)} = \left| k^2 x^{-(2k+2)} - k(k+1)x^{-(k+2)} \right| \sim k^2 x^{-(2k+2)} \text{ as } x \to 0,$$

$$f^*(2 \mid \rho''(x) \mid / \rho(x)) \sim p k^{2(1-1/p)} x^{-(2k+2)(p-1)/p} \exp\left(k^{2/p} x^{-(2k+2)/p} \right).$$

Accordingly, $\rho f^*(2 \mid \rho''(x) \mid / \rho) \in L^1(\Omega)$ if and only if $k > (2k+2)/p$, or, equivalently, $k > 2/(p-2)$.

- The case when $f \sim |u|(\log(1+|u|))^\gamma$ with $\frac{3}{2} \leq \gamma \leq 2$ is open.
- If $f = f(x,u)$ satisfies the good-sign condition $f(x,u)u \geq 0$ the nonlinearity is dissipative but data are controllable after some waiting time. ([Anita-Tataru '02])

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Setting of the problem

Consider the viscous Hamilton-Jacobi equation

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = v\,\chi_\omega & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } \Sigma := (0,T)\times\partial\Omega \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

where

- $\Omega \subset \mathbf{R}^N$ is a bounded open set
- the control is localized in an open subset $\omega \subset\subset \Omega$
- the control $v \in L^\infty((0,T)\times\omega)$
- $u_0 \in C_0(\overline{\Omega})$
- $q > 1$ (superlinear case)

Goal: (Null controllability)
Find a control $v \in L^\infty((0,T)\times\omega)$ such that $u(T) = 0$.

Motivation

- Existing methods only ensure the controllability of small solutions.
- For nonlinearities of the form $f(|u|)$ it is easy to guess that control may fail for large data because of blow up phenomena.
- But $f(|\nabla u|)$ does not necessarily become large when u is blowing-up.
- So ...?

In our analysis of the problem

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = v\,\chi_\omega & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } \Sigma := (0,T)\times\partial\Omega \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

- both the cases $q \leq 2$ and $q > 2$ are included
- no sign condition is assumed neither on the solutions nor on u_0. In particular, the sign of the nonlinearity does not play any role (and can be reversed by changing u into $-u$).

Recall:

(i) If $q \leq 2$, $\exists$ a unique weak solution $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$ which exists globally in time and is smooth for $t > 0$.

(ii) If $q > 2$, but $u_0 \in C(\overline{\Omega})$, $\exists$ a unique viscosity solution which is global in time [Barles-Da Lio] (boundary data are taken in a relaxed sense, as in first order problems)

What about the general case $f = f(x, \nabla u)$?

Here the blow-up is prevented by maximum principle.

$$u_t - \Delta u + |\nabla u|^q = 0 \quad \Rightarrow \quad \Big|u(t)\Big|_\infty \leq \Big|u_0\Big|_\infty$$

independently of the sign of the nonlinearity!
Overall, combining the effects of superlinear growth and intrinsic dissipation of the system, we prove the following type of results:

1. Any initial data u_0 can be controlled in some time $T(u_0)$
2. A waiting time may actually exist; for any given $T > 0$, there exist initial data u_0 which cannot be controlled at time T.
3. Asymptotic rates of the time of control when the L^∞-norm of u_0 goes to zero or infinity.

Given

$$(P)\ \begin{cases} u_t - \Delta u + |\nabla u|^q = v\,\chi_\omega & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } \Sigma := (0,T)\times\partial\Omega \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

we define

$$T(u_0) := \inf\,\{t>0:\ \text{(P) can be controlled at zero at time } t\}$$

$$T(r) = \sup\left\{T(u_0)\,,\ \left|u_0\right|_\infty \le r\right\}$$

Theorem

We have $0 < T(r) < \infty$ for every $r > 0$, and there exist $\kappa, K, \lambda, \Lambda$ (only depending on q, Ω, ω) such that

(i)(small data) $\dfrac{\kappa}{\ln\left(\frac{1}{r}\right)} \le T(r) \le \dfrac{K}{\ln\left(\frac{1}{r}\right)}$ *as* $r \to 0^+$

(i)(large data) $\lambda\, r \le T(r) \le \Lambda\, r$ *as* $r \to \infty$.

- The need of **waiting time** is similar to other dissipative systems: Burgers' equation ($1-d$-case, see [Fernández Cara-Guerrero, 2007]), zero order nonlinearities with good sign [Anita-Tataru, 2002].
- The $\log(1/r)$ rate for small initial data also holds in those cases and it is related the rate of the observability constant for the heat equation $\sim e^{c_0/T}$.
- The key idea is to exploit the strong nonlinearity to localize the energy estimates away from the region where the control is localized.

Proof of the finite-time control

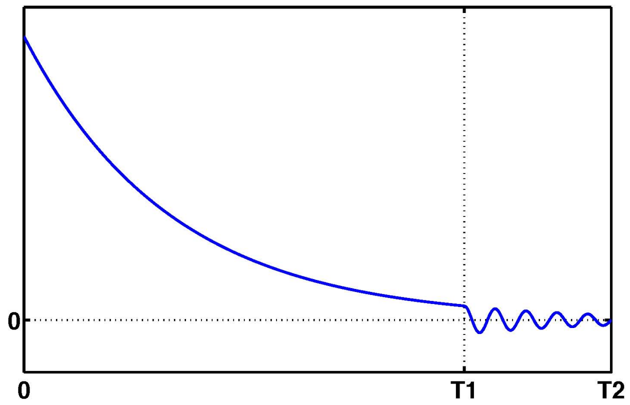
It relies on two basic steps:

1. Control of small data by linear estimates and fixed point argument around $u \equiv 0$ and $v \equiv 0$.
2. The system free of control naturally leads the solution to become small & smooth.

Lemma

In the absence of control $\exists K, \lambda, C$ (only depending on q, Ω):

$$\left|u(t)\right|_\infty + \left|\nabla u(t)\right|_\infty \leq Ce^{-\lambda t} \quad \forall t \geq K \left|u_0\right|_\infty$$


0
0
T1
T2

Proof of the decay rate

Step 1: **Bernstein type gradient estimates.**

$$||\nabla u||_{\infty} \leq C(||u_0||_{\infty}/t)^{1/q}.$$

Step 2: **Decay.**

Linearize the nonlinear equation and use the former decay on the nonlinear term to use comparison arguments for the linearized equation.

Barrier functions

The lower bounds on the waiting time depend on local estimates and the existence of universal local barriers ([Lasry-Lions '89]).

Given any $\omega_0 \subset \Omega \setminus \omega$,

$$\exists\ U\ :\quad \begin{cases} U_t - \Delta U + |\nabla U|^q = 0 & \text{in} \quad (0,T) \times \omega_0 \\ U \to +\infty & \text{as} \quad x \to \partial\omega_0\,, \\ U(0) = 0 \end{cases}$$

Roughly speaking: if $u_0 < 0$ in some $\omega_0 \subset \Omega \setminus \omega$, then $u < U - \varepsilon$, so u needs some time to get at zero, *only depending on the shape of U*, regardless of the control v acting on ω.

Rmk: More generally, such universal barriers exist for the equation

$$u_t - \Delta u + h(|\nabla u|) = 0$$

provided

$$\int^{+\infty} \frac{ds}{h(s)} < \infty.$$

A generalization: Control to trajectories

Given a free trajectory $\hat{u}$ it is also possible to control the viscous H-J so that $u(T) = \hat{u}(T)$.

The control time is estimated in terms of $\left| u_0 - \hat{u}_0 \right|_\infty$.

An open problem about waiting time

The waiting time certainly exists if $u_0 < 0$ in some $\omega_0 \subset \Omega \setminus \omega$.

Can the waiting time be avoided if $u_0 \geq 0$?

This is related to the sign of the nonlinearity: in the equation

$$u_t - \Delta u + |\nabla u|^q = v\chi_\omega$$

the nonlinear term is dissipative in case of positive solutions.

Similar open problems for semi linear equations

$$u_t - \Delta u + f(u) = v\chi_\omega\,, \quad \text{with } f(s)s \geq 0$$

both in both concerns the need of waiting time or the impossibility of controlling blowing-up.

- **An open problem:** What about the case of odd nonlinearities (even in $1-d$)?

$$u_t - \Delta u + |\nabla u|^{q-1}\nabla u = v\,\chi_\omega \text{in } (0,T)\times\Omega.$$

- **Another one:** Letting viscosity tend to zero!
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  - Tatsien, Li. Exact boundary controllability for quasilinear wave equations. J. Comput. Appl. Math. 190 (2006), no. 1-2, 127Ð135.
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