Control of semilinear waves

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ESAIM:COCV, 1999: Cannarsa, Komornik & Loreti

ESAIM: Control, Optimisation and Calculus of Variations April 1999, Vol. 4, p. 37–56
URL: http://www.emath.fr/cocv/

WELL POSEDNESS AND CONTROL OF SEMILINEAR WAVE EQUATIONS WITH ITERATED LOGARITHMS*

PIERMARCO CANNARSA[1], VILMOS KOMORNIK[2] AND PAOLA LORETI[3]

Abstract. Motivated by a classical work of Erdős we give rather precise necessary and sufficient growth conditions on the nonlinearity in a semilinear wave equation in order to have global existence for all initial data. Then we improve some former exact controllability theorems of Imanuvilov and Zuazua.

Résumé. Motivé par un travail classique d'Erdős on donne des conditions nécessaires et suffisantes de croissance de la non linéarité dans une équation des ondes semilinéaire pour l'existence des solutions globales pour toutes les données initiales. Ensuite on améliore certains théorèmes antérieurs de contrôlabilité exacte de Imanuvilov et de Zuazua.

Roughly they establish the controllabililty of the $1-d$ semilinear wave equation for nonlinearities of the form

$$f(s) = s[\log[\log[\log \ldots |s|]]]^2$$

Proof [1]

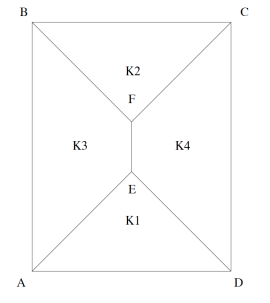


Figure 2 in their paper

[1] Similar sidewise energy arguments have been used in other contexts such as waves on networks: J. Lagnese, G. Leugering and G. Schmidt...

Why?

The nonlinearity

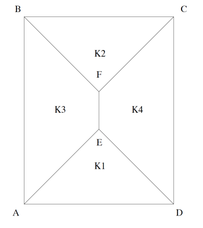
$$f(s) = s[\log[\log[\log \dots |s|]]]^2$$

is *subcritical* both for

$$y_{tt} - y_{xx} + f(y) = 0$$

and

$$y_{xx} - y_{tt} - f(y) = 0$$



Osgood condition

For first order ODEs

$$x'(t) = f(x(t))$$

the criticality condition is

$$\int_{z_0}^{\infty} \frac{1}{f(z)} dz = \infty.$$

This condition ensures global well-posedness, while finite-time blow up occurs if the integral converges.

Thus

$$f(s) = s[\log[\log[\log \ldots |s|]]]$$

is **critical**.

2nd order Osgood condition

For second order ODEs

$$x''(t) = f(x(t))$$

multiplying by $x'(t)$ we get

$$\frac{1}{2}|x'(t)|^2 - F(x(t)) = C$$

And therefore

$$x'(t) = \pm\sqrt{2(C + F(x(t))},$$

the critical threshold becomes

$$f(s) = s[\log[\log[\log \ldots|s|]]]^2.$$

Multi-d

The multi-d analog of this result is unknown, because of the ill-posedness of the sidewise wave equation.
A partial result is known for non-linearities of the order of [2]

$$|f(s)| << |s| \log^{1/2}(|s|)|, \text{ as } |s| \to \infty.$$

Employing techniques based on Carleman inequalities one could expect to extend this result to the range

$$|f(s)| << |s| \log^{3/2}(|s|)|, \text{ as } |s| \to \infty,$$

as it occurs for the heat equation.

But getting to the 1-d range seems hard although nothing excludes the result to be true, as far as we know.

[2] X. Zhang and E. Z. Exact controllability of the semi-linear wave equation. In "Unsolved problems in mathematical systems and control theory", Princeton University Press, 2004, pp. 173-178.

This $2/3$ exponent may not be improved through Carleman estimates as pointed out by Th. Duyckaerts, X. Zhang and EZ (Annales IHP, 2005), based on the following result by V. Z. Meshkov, 1991.

Theorem

(Meshkov, 1991). *Assume that the space dimension is $n = 2$. Then, there exists a nonzero complex-valued bounded potential $q = q(x)$ and a non-trivial complex valued solution $u = u(x)$ of*

$$\Delta u = q(x)u, \qquad \text{in } \mathbb{R}^2, \tag{1}$$

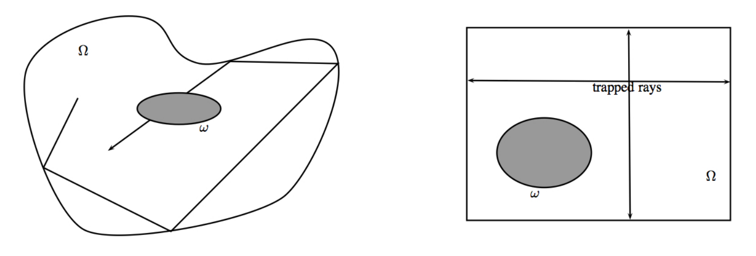
with the property that

$$| u(x) | \leq C \exp(- | x |^{4/3}), \qquad \forall x \in \mathbb{R}^2 \tag{2}$$

for some positive constant $C > 0$.

What if the control time is too short?

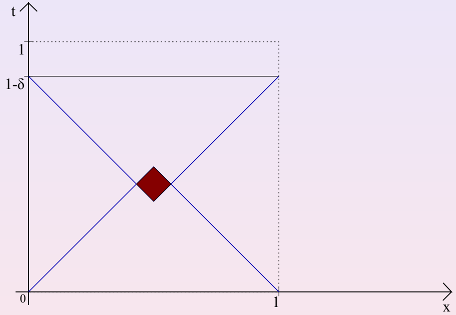
These results hold for large enough time horizons of control. Roughly, under the so-called Geometric Control Condition (GCC) by Bardos - Lebeau - Rauch (1988). It asserts, roughly, that all rays of geometric optics enter the control set ω in time T.



When this condition holds all data can be controlled.
But what when T is too short for GCC to be fulfilled?

1-d : $T < 2$

In 1-d if $T < 2$ the space of controllable initial data can be characterised either using characteristic arguments or Fourier series expasions:[3] [4]



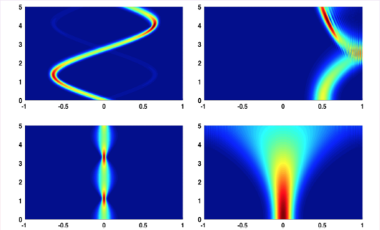
[3] J. Lohéac and E. Zuazua, Norm saturating property of time optimal controls for wave-type equations, CPDE 2016, Bertinoro, 2016, IFAC-PapersOnLine, 49 (8) 37-42, 2016.

[4] D. A. Ivanov, M. M. Potapov. Approximations to Time-Optimal Boundary Controls for Weak Generalized Solutions of the Wave Equation. Computational Mathematics and Mathematical Physics, 2017, Vol. 57, No. 4, pp. 607-625.

Numerical algorithms

How to build numerical algorithms enabling to compute the minimal control time for a specific initial datum?

Note that, so far, all effort have been devoted to build numerical algorithms enabling to get the GCC. But nothing has been done for shorter T's.[5], [6]



[5] E. Z. Propagation, observation, and control of waves approximated by finite difference methods. SIAM Review, 47 (2) (2005), 197-243.

[6] A. Marica and E. Zuazua, Propagation of $1-D$ waves in regular discrete heterogeneous media: A Wigner measure approach, Found. Comput. Math. (2015) 15:1571-1636.